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MATHEMATICS

A Limit Theorem for Non-decreasing Random Functions

by

M. FISZ

Presented by H. STEINHAUS on May 4, 1958

We consider a sequence $\{Y_k(t), -\infty < t < \infty\}$ $(k = 1, 2, ...)$ of real, separable and independent stochastic processes, defined on the probability space (Ω, B, P), where B is the Borel field of ω sets $(\omega \in \Omega)$ with respect to which all the $Y_k(t)$ $(-\infty < t < \infty;\ k = 1, 2, ...)$ are measurable, and P is the probability measure on B. The processes $Y_k(t)$ are assumed to be equally distributed, i. e. for each finite set of parameter points $t_1, ..., t_s$ the random vectors $\{Y_k(t_1), ..., Y_k(t_s)\}$ $(k = 1, 2, ...)$ are equally distributed. Denote:

$$EY_k(t) = F(t)\,,$$

$$X_n(t) = \frac{1}{n}\sum_{k=1}^{n} Y_k(t)$$

$$Z_n(t) = |X_n(t) - F(t)|\,,$$

$$D_n = \sup_{-\infty<t<\infty} Z_n(t)\,.$$

The measurability of D_n (Doob [1] p. 53) follows from the assumed separability of the processes considered.

THEOREM. *Let* $\{Y_k(t), -\infty < t < \infty\}$ $(k = 1, 2, ...)$ *be a sequence of real, separable, independent and equally distributed stochastic processes. If the relation* $P(y_k(t_1) \leqslant y_k(t_2)) = 1$ *holds for arbitrary* $t_1, t_2 (t_1 < t_2)$ *and if* $F(t)$ $(-\infty < t < \infty)$ *exists and satisfies the equalities*

$$F(-\infty) = b; \quad F(+\infty) = c\,, \tag{1}$$

where b *and* c *are some constants, then*

$$P(\lim_{n\to\infty} D_n = 0) = 1\,. \tag{2}$$

Proof. Let the assumptions of the Theorem be satisfied. It follows that $F(t)$ is a non-decreasing function. Assume — since this is not an

essential restriction – that $c-b$ is a positive integer. Denote for $r = 1, 2, 3, \ldots$ by $t_{r,j} (j = 1, \ldots, (c-b)r)$ the smallest value of t satisfying the double inequality

$$F(t-0) \leqslant b + \frac{j}{r} \leqslant F(t+0) . \tag{3}$$

Denote further by $A_r (r = 1, 2, \ldots)$ the event

$$\left(\lim_{n\to\infty} \max_{1\leqslant j\leqslant (c-b)r} \max[Z_n(t_{r,j}-0), Z_n(t_{r,j}), Z_n(t_{r,j}+0)] = 0\right)$$

and

$$A = \prod_{r=1}^{\infty} A_r .$$

Since the realizations of $Y_k(t)$ are, with probability 1, monotonic functions we obtain, in virtue of Lebesgue's theorem,

$$EY_k(t-0) = F(t-0)$$

$$EY_k(t+0) = F(t+0) .$$

Kolmogorov's law of large numbers [3] implies then

$$P(A) = 1 . \tag{4}$$

We remark now that the relations

$$P(X_n(t_{r,j}+0) \leqslant X_n(t) \leqslant X_n(t_{r,j+1}-0);\ t_{r,j} < t < t_{r,j+1}) = 1 , \tag{5}$$

$$F(t_{r,j}+0) \leqslant F(t) \leqslant F(t_{r,j+1}-0) \qquad (t_{r,j} < t < t_{r,j+1}) \tag{6}$$

are evidently true. Relation (3) implies for $t_{r,j} < t_{r,j+1}$ the double inequality

$$0 \leqslant F(t_{r,j+1}-0) - F(t_{r,j}+0) \leqslant \frac{1}{r} . \tag{7}$$

Formulae (5)–(7) give at once for $j = 1, \ldots, (c-b)r-1$

$$P\left(\sup_{t_{r,j}<t<t_{r,j+1}} Z_n(t) \leqslant \max[Z_n(t_{r,j}+0), Z_n(t_{r,j+1}-0)] + \frac{1}{r}\right) = 1 . \tag{8}$$

Denote

$$L_{nr} = \max[\sup_{t_{r,1}\leqslant t\leqslant t_{r,(c-b)r}} Z_n(t), Z_n(t_{r,1}-0), Z_n(t_{r,(c-b)r}+0)] .$$

From (8) we obtain

$$P\left(L_{nr} \leqslant \max_{1\leqslant j\leqslant (c-b)r} \max[Z_n(t_{r,j}-0), Z_n(t_{r,j}), Z_n(t_{r,j}+0)] + \frac{1}{r}\right) = 1 . \tag{9}$$

The last relation implies

$$P(\lim_{n\to\infty} \sup_{-\infty<t<\infty} Z_n(t) = 0) \geqslant$$

$$P(\lim_{n\to\infty} \lim_{r\to\infty} \max_{1\leqslant j\leqslant (c-b)r} \max[Z_n(t_{r,j}-0), Z_n(t_{r,j}), Z_n(t_{r,j}+0)] = 0) \geqslant P(A) .$$

In virtue of (4) we finally obtain the assertion of the Theorem.

The special case of the Theorem proved with $Y_k(t)$ given by the formula

$$Y_k(t) = \begin{cases} 0 & (t \leqslant T_k) \\ 1 & (t > T_k)\,, \end{cases}$$

where $\{T_k\}$ is a sequence of independent random variables with a common distribution function $F(t)$, has been given by Glivenko [2].

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MATHEMATICS

The System of Axioms of the Möbius Geometry

by
A. JANKOWSKI

Presented by A. MOSTOWSKI on May 8, 1958

In [3] L. Dubikajtis has presented a system of axioms, which are satisfied in various geometries (among others in Möbius geometry). The author asks how to complete this system in order to obtain categorical systems of axioms for these geometries.

The purpose of this note is to give the solution of the above problem in the case of n-dimensional Möbius geometry.

Let us first define an n-dimensional Möbius geometry by its model M_n in metric Euclidean geometry.

Let E^n be an n-dimensional Euclidean point-space. If we adjoin to the set of all points of this space a new element, which will be called the point at infinity, and assume that this element lies in each line and does not lie in any sphere, then such a set will be called an n-dimensional Möbius space.

Each of the following subsets of E^n:

1. hypersphere, i. e. the surface of n-dimensional sphere,
2. hyperplane, i. e. $(n-1)$-dimensional plane

will be called the M-sphere.

The points of Möbius space will be denoted by capital Latin letters and M-spheres by small Latin letters.

We shall say that point P is incident with M-sphere p, if either

1. P belongs to E^n and P is incident in the sense of metric geometry with M-sphere p, or
2. P is the point at infinity, and p is an arbitrary hyperplane.

The incidence relation will be denoted by "$-$". Then we shall write $P-p$ or $p-P$. The negation of this relation will be denoted by "$\nmid$".

The model M_n consists of:

1. the set of points of the Möbius space
2. the set of M-spheres
3. the incidence relation.

The group of all one to one transformations which map each point of Möbius space into a point of Möbius space, and each M-sphere into an M-sphere, while the incidence relation is preserved, will be called the group of Möbius transformations.

This group defines the Möbius geometry.

The system of axioms will be based on the following primitive notions:

1. the set of points,
2. the set of M-spheres,
3. the incidence relation.

The symbol $(P_1, ..., P_k)-(p_1, ..., p_k)$ means that P_i-p_j for $i = 1, ..., l$; $j = 1, ..., k$. If $l = k$, or $l = 1$, then the parantheses can be omitted.

Let us define the following logical notions:

DEFINITION 1. We shall call the points $P_1, ..., P_k$ dependent on $Q_1, ..., Q_l$, if for each M-sphere p, such that $p-(Q_1, ..., Q_l)$, we have $p-(P_1, ..., P_k)$. This relation will be denoted by $(P_1, ..., P_k)d(Q_1, ..., Q_l)$.

DEFINITION 2. We shall call the points $(P_1, ..., P_k)$ dependent, if there exists such an i that $P_i d(P_1, ..., P_{i-1}, P_{i+1}, ..., P_k)$. This relation will be denoted by $(P_1, ..., P_k)_d$. In the opposite case these points are independent. This will be denoted by $(P_1, ..., P_k)_i$.

The dependence and independence of M-spheres is defined in an analogous way.

DEFINITION 1′. $(p_1, ..., p_k)d(q_1, ..., q_l)$ if, for each point P such that $P-(q_1, ..., q_l)$, we have $P-(p_1, ..., p_k)$.

DEFINITION 2′. We shall call M-spheres $p_1, ..., p_k$ dependent, if there exists an i such that $p_i d(p_1, ..., p_{i-1}, p_{i+1}, ..., p_k)$. In the opposite case, the M-spheres are independent.

DEFINITION 3. We shall call M-spheres, a_1 and a_2 tangent, if there exists exactly one point A, such that $A-(a_1, a_2)$. This relation will be denoted by $a_1 \ominus a_2$.

DEFINITION 4. We shall call the M-spheres a_1 and a_2 connected, if there exist points A_1 and A_2 such that

1. $(A_1, A_2)_i$
2. $(A_1, A_2)-(a_1, a_2)$.

DEFINITION 5. We shall say that the points A_1 and A_2 are separated by an M-sphere a, if a is not incident with A_1 and A_2 simultaneously, and if a is connected with each M-sphere passing through these points. The separation will be denoted by $A_1|a|A_2$.

DEFINITION 6. We shall call the points $A_1, ..., A_k$ conjugate with M-spheres $a_1, ..., a_k$, if:

1. A_i-a_i; $i = 1, ..., k$

2. for arbitrary $i = 1, \ldots, k$; $j = 1, \ldots, k$ there exists an M-sphere s_{ij} such that

$$\begin{array}{ccccc} a_i & \ominus & s_{ij} & \ominus & a_j \,. \\ & \diagdown\,\diagup & & \diagdown\,\diagup & \\ & A_i & & A_j & \end{array}$$

This relation will be denoted by $(A_1, \ldots, A_k)s(a_1, \ldots, a_k)$.

Definition 7. We shall say that the points A_1, A_2 are separated by A_3, A_4, if A_1, A_2 are separated by each M-sphere passing through A_3, A_4 and not passing through A_1, A_2. This relation will be denoted by $A_1A_2|A_3A_4$.

Making use of these notions we shall introduce a system of axioms, which will be called the system of axioms A_n.

These axioms are divided into 6 groups according to the notions which appear in them. The first three axioms and definitions $1, 2, 1', 2'$ are taken from [3].

I. The axioms of dependence.

AI. 1. If $(A_1, \ldots, A_k)_i - (a_1, \ldots, a_l)_i$; $k + l = n + 2$, $A_0 - (a_1, \ldots, a_l)$, $(A_1, \ldots, A_k) - b_0$, then $A_0 - b_0$.

AI. 2. If $(A_1, \ldots, A_k)_i - (a_1, \ldots, a_l)_i$; $k + l = n + 2$, then there exists

1. such A_{k+1} that $(A_1, \ldots, A_k, A_{k+1})_i - (a_1, \ldots, a_l)_i$,

and

2. such a_{l+1} that $(A_1, \ldots, A_k)_i - (a_1, \ldots, a_l, a_{l+1})_i$.

AI. 3. There exist two independent points.

II. The axioms of tangence.

AII. 1. If $A_1 - y \neq A_2$, then there exists such y_1 that $y_1 \ominus y$ and $y_1 - -(A_1, A_2)$.

AII. 2. If $y_1 - A \neq y_2$, then there exist exactly two M-spheres x_1, x_2 such that the following conditions are satisfied: $A - (x_1, x_2)$ and $y_1 \ominus x_i \ominus y_2$ for $i = 1, 2$.

AII. 3. If there exists such point A that $A - (y_1, y_2, y_3)$ and $y_1 \ominus y_2 \ominus y_3$, then $y_1 \ominus y_3$.

III. The axioms of separation.

AIII. 1. For each three different points A_1, A_2, A_3 there exists an M-sphere y such that $A_1 - y$ and $A_2|y|A_3$.

AIII. 2. If $A_1|y|A_2$ and $A_3 \neq y$, then either $A_1|y|A_3$ or $A_2|y|A_3$.

AIII. 3. If $A_1|y|A_2$ and $B_1 - y$, then there exists a point B_2 such that $B_2 - y$ and $A_1A_2|B_1B_2$.

IV. The axiom of connection.

AIV. 1. If A_1, A_2, A_3, x are such that

1. $(A_1, A_2, A_3)_i$

and

2. each y satisfying the condition $y-(A_1, A_2, A_3)$ is connected with x,

then there exist exactly two points B_1, B_2 such that $(B_1, B_2)-x$ and $(B_1, B_2)d(A_1, A_2, A_3)$.

V. The axioms of conjugation.

AV. 1. If B_1-y_1, B_2-y_2 and A_1, A_2, B_1, B_2 are different points and

1. $(A_1, A_2, B_1, B_2)_d$
2. $(A_1, A_2)s(y_1, y_2)$

then $(B_1, B_2)s(y_1, y_2)$.

AV. 2. If each among three different M-spheres y_1, y_2, y_3 is dependent on two others, and $(A_1, A_2)s(y_1, y_2)$, then there exist two different points B_1 and B_2 such that $(A_1, A_2, B_1)s(y_1, y_2, y_3)$ and $(A_1, A_2, B_2)s(y_1, y_2, y_3)$.

AV. 3. If $(A_1, A_2, B_2)s(y_1, y_2, y_3)$ and there exists such point A that $A-(y_1, y_2, y_3)$ then $(y_1, y_2, y_3)_a d$.

VI. The axiom of continuity.

AVI. 1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two sets of points such that there exist points A_0 and B_0 satisfying the condition A_0B/AB_0 for arbitrary $A \in \mathfrak{A}, B \in \mathfrak{B}$. Then there exists such point C that B_0C/AB.

Now let us outline the proof that the system of axioms A_n for $n \geqslant 2$ is the system of axioms of n-dimensional Möbius geometry defined above.

In order to prove this we shall show that:

1. the axioms are satisfied in the model M_n,
2. the system of axioms A_n is categorical.

The first part of this proof is not difficult and rests on the properties of Möbius transformations given in [1] and [2].

The second part of this proof is analogous to the argumentation given in [4].

Definition 8. By a k-dimensional subspace we understand:

1. the empty set, in the case $k = -2$,
2. one point, in the case $k = -1$,
3. the set of all points dependent on the set of $k+2$ independent points, in the case $0 \leqslant k \leqslant n-1$,
4. the set of all points, if $k = n$.

The subspaces will be denoted by small Latin letters, whose upper indices denote dimension. The dimension of subspace a^k will be denoted by $D(a^k)$. It can be proved that the set of subspaces consitutes a lattice if a partial order is given by the ordinary inclusion and the operations are defined as follows:

1. $a^k \cup b^l$ is the last subspace containing a^k and b^l,
2. $a^k \cap b^l$ is the largest subspace contained in a^k and b^l.

This lattice has the following properties:

1. If the intersection of a and b contains two different points, then $D(a^k \cup b^l) + D(a^k \cap b^l) = k+1$.

2. Between the set of M-spheres and the set of $(n-1)$-dimensional subspaces we can define one to one correspondence in such a manner that, to each subspace a^{n-1}, there corresponds an M-sphere satisfying the condition $y - a^{n-1}$. Moreover, if a^{n-1} corresponds to y, then, for each point B, the conditions $B \epsilon a^{n-1}$ and $B - y$ are equivalent.

DEFINITION 9. The subspace a^k will be called tangent to b^l if either

1. $a^k \subset b^l$

or

2. $k \leqslant l$; $D(a^k \cap b^l) = -1$; $D(a^k \cup b^l) = l+1$.

The tangency of subspaces will be denoted by $a^k t b^l$.

Let us consider an arbitrary fixed point, which will be denoted by I and will be called the point at infinity. Now we can define (in definitions 10-12) some relative notions (depending on the choice of I).

DEFINITION 10. Subspaces passing through point I will be called "planes" and the remaining ones – "elipsoids". The $(n-1)$-dimensional subspaces will be called "hyperplanes" and "hyperspheres", respectively.

DEFINITION 11. Point C lies between points A and B if $AB|CI$.

DEFINITION 12. "Planes" are parallel, if they are tangent.

It is easy to see that all axioms of affine geometry are satisfied by the set of points, "planes" and the two above relations. Besides, the elements called here "hyperspheres" are, in this geometry, $(n-1)$-dimensional similar elipsoids. In the affine geometry, metric can be introduced by taking a $(n-1)$-dimensional elipsoid as a hypersphere. Then all elipsoids similar to that hypersphere are also hyperspheres, and the set of these elipsoids is the set of all hyperspheres of the metric space thus obtained.

Now let $\mathfrak{M}_n$ and $\mathfrak{M}_n^*$ be two models of the metric geometry defined as above over the system of axioms A_n. From the fact that the metric geometry is categorical it follows that there exists isomorphism between these two models of metric geometry. Under this isomorphism the proper points, hyperplanes and hyperspheres of the first model correspond to the proper points, hyperplanes and hyperspheres of the second model. The correspondence is such that the incidence relations are preserved. This isomorphism can be easily extended by letting the point at infinity of the first model correspond to the point at infinity of the second model. The incidence of this point will also be preserved. Thus we obtain the isomorphism of two arbitrary models of the system of axioms A_n. This means that the system is categorical.

Thus we have completed an outline of the proof of the following theorem:

If $n \geqslant 2$, *then the system of axioms* A_n *is the system of axioms of the n-dimensional Möbius geometry.*

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MATHÉMATIQUE

Propriété limite du potentiel généralisé de simple couche relativement à l'équation parabolique normale

par

H. MILICER-GRUŻEWSKA

Présenté par T. WAŻEWSKI, le 22 mai, 1958

Le potentiel généralisé de simple couche [1] relativement à l'équation parabolique normale:

(i)
$$\hat{\Psi}_t(u) = \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(A, t)\, u''_{x_\alpha x_\beta} + \sum_{\alpha=1}^{n} b_\alpha(A, t)\, u'_{x_\alpha} + c(A, t)\cdot u - \frac{\partial u}{\partial t} = 0\,, \quad A \in \Omega$$

c'est l'intégrale de surface de la forme suivante:

(ii)
$$U(A, t) = \int_0^t \int\!\!\int_s \Gamma(A, t;\, Q, \tau)\varphi(Q, \tau)\, dQ\, d\tau\ ^*)\,,$$

où $\Gamma(A, t;\, B, \tau)$, $A \neq B$, A, $B \in \Omega$, désigne la solution fondamentale de l'équation (i), $\varphi(Q, \tau)$ la densité, Ω est un domaine dans l'espace euclidien à n-dimensions ($n \geqslant 3$), limité par la surface fermée S.

W. Pogorzelski m'a proposé d'examiner la limite de l'intégrale (ii) pour $t \to \infty$. Ce problème fut déjà traité par Eidelman [2] et M. Krzyżański [3], mais dans l'hypothèse que les coefficients de l'équation parabolique sont dérivables, ou dans un cas spécial. Ici on admettra seulement que les coefficients de l'équation (i) sont hölderiens, ainsi que leurs limites, qui existent uniformément, de même que la limite de la densité $\varphi(Q, t)$ pour $t \to \infty$.

Je prouve dès lors que la limite du potentiel en question est le potentiel généralisé de simple couche de la densité limite $\bar{\varphi}(Q) = \lim_{t\to\infty} \varphi(Q, t)$

---

*) dQ désigne l'élément d'aire de l'hypersurface S au point Q, pareillement dB désignera l'élément de volume du domaine Ω au point B.

relat. à l'équation élliptique normale. Les coefficients de cette équation sont les limites des coefficients correspondants de l'équation parabolique. Cette équation elliptique sera nommée équation limite de l'equation (i).

Mais pour que le passage à la limite soit légitime il faut admettre que le diamètre du domaine Ω n'est pas trop grand. C'est d'ailleurs d'accord avec la question que soulève l'existence et la forme de la solution fondamentale de l'équation elliptique, [4] et [5].

On admet, conformément à l'introduction, les hypothèses suivantes:

I. Les fonctions $a_{\alpha\beta}(A, t)$ sont définies dans la région fermée Ω' contenant $\Omega + S$ pour toute valeur non négative du parametre t, donc pour:

$$(1) \qquad A(x_1, x_2, \ldots, x_n) \in \Omega', \quad t \geqslant 0, \quad \Omega' \supset \Omega + S$$

et on a uniformément dans Ω':

$$(1') \qquad \lim_{t\to\infty} a_{\alpha\beta}(A, t) = \bar{a}_{\alpha\beta}(A);$$

en outre nous admettons l'inégalité d'Hölder

$$(1'') \qquad |a_{\alpha\beta}(A, t) - a_{\alpha\beta}(A_1, t_1)| \leqslant \text{const}[r_{AA_1}^h + |t - t_1|^{h'}]; \qquad A, A_1 \in \Omega',$$

d'où résulte que:

$$(1''') \qquad |\bar{a}_{\alpha\beta}(A) - \bar{a}_{\alpha\beta}(A_1)| \leqslant \text{const} r_{AA_1}^h; \qquad A, A_1, \in \Omega',$$

pour $\alpha, \beta = 1, 2, \ldots n$, où r_{AA_1} désigne la distance des points A et A_1, h et h' sont des constantes et $0 < h \leqslant 1, 0 < h' \leqslant 1$.

II. Les fonctions $b_\alpha(A, t)$ et $c(A, t)$ sont continues par rapport à la variable $t \geqslant 0$, vérifient par rapport aux variables spatiales la condition d'Hölder, et on a uniformément dans Ω':

$$(2) \qquad \lim_{t\to\infty} b_\alpha(A, t) = \bar{b}_\alpha(A), \lim_{t\to\infty} c(A, t) = \bar{c}(A), \qquad A \in \Omega';$$

en outre on a

$$|b_\alpha(A, t) - b_\alpha(A_1, t)| \leqslant \text{const} r_{AA_1}^h; \; |c(A, t) - c(A_1, t)| \leqslant \text{const} r_{AA_1'}^h,$$

$$|\bar{b}_\alpha(A) - \bar{b}_\alpha(A_1)| \leqslant \text{const} r_{AA_1}^h; \; |\bar{c}(A) - \bar{c}(A_1)| \leqslant \text{const} r_{AA_1}^h$$

$$0 < h \leqslant 1.$$

III. La forme quadratique

$$(3) \qquad \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(A, t) X_\alpha X_\beta$$

est définie positive dans la région fermée Ω', $t \geqslant 0$.

IV. La fonction $\varphi(Q,t)$ est définie, continue, bornée et intégrable sur la surface S, pour chaque valeur positive du parametre t, et on a uniformément sur S

$$\lim_{t\to\infty}\varphi(Q,t)=\bar{\varphi}(Q)\,,\qquad Q\in S\,. \tag{4}$$

Il résulte que les fonctions $\bar{\varphi}(Q)$, $\bar{a}_{\alpha\beta}(A)$, $\bar{b}_{\alpha}(A)$, $\alpha,\beta=1,\ldots,n$ et $\bar{c}(A)$ sont continues et bornées sur la surface S resp. dans le domaine Ω'.

V. La forme quadratique

$$\sum_{\alpha,\beta=1}^{n}\bar{a}_{\alpha\beta}(A)X_{\alpha}X_{\beta} \tag{3'}$$

est définie positive dans la région Ω' *).

Si nous désignons par $a^{\alpha\beta}(A,t)$ et $\bar{a}^{\alpha\beta}(A)$ les éléments des matrices inverses des matrices $\|a_{\alpha\beta}(A,t)\|$ resp. $\|\bar{a}_{\alpha\beta}(A)\|$, nous pouvons écrire

$$\lim_{t\to\infty}a^{\alpha\beta}(A,t)=\bar{a}^{\alpha\beta}(A),\ \alpha\beta=1,2\ldots n\qquad A\in\Omega' \tag{5}$$

uniformément dans Ω'.

Soit maintenant pour $A=A(x_1,\ldots,x_n)$, $B=B(\xi_1,\ldots,\xi_n)$:

$$\vartheta^{M,t}(A,B)=\sum_{\alpha,\beta=1}^{n}a^{\alpha\beta}(M,t)(x_{\alpha}-\xi_{\alpha})(x_{\beta}-\xi_{\beta})\,, \tag{6}$$

$$\bar{\vartheta}^{M}(A,B)=\sum_{\alpha,\beta=1}^{n}\bar{a}^{\alpha\beta}(M)(x_{\alpha}-\xi_{\alpha})(x_{\beta}-\xi_{\beta})\,, \tag{6'}$$

alors on a, en vertu de (5), uniformément dans Ω'

$$\lim_{t\to\infty}\vartheta^{M,t}(A,B)=\bar{\vartheta}^{M}(A,B)\,,\qquad M,A,B\in\Omega'\,. \tag{7}$$

Rappellons [1] que la solution fondamentale Γ de l'équation (i) est donnée par les formules suivantes

$$\Gamma(A,t;\,B,\tau)= \tag{8}$$

$$=w^{B,\tau}(A,t;\,B,\tau)+\int_{\tau}^{t}\iiint_{\Omega'}w^{M,\theta}(A,t;\,M,\Theta)\Phi(M,\Theta;\,B,\tau)\,dM\,d\Theta\,,\ A,B\in\Omega\,,$$

où la fonction

$$w^{B,\tau}(A,t;\,B,\tau)=(t-\tau)^{-n/2}\exp[-\vartheta^{B,\tau}(A,B)/4(t-\tau)]\,,\qquad t>\tau>0 \tag{8'}$$

est la bien connue *quasi-solution*, et la fonction Φ est la solution d'une équation intégrale de Volterra de noyau N, qui est donnée par la formule

$$N(A,t;\,M,\Theta)= \tag{8''}$$

$$\lambda(A,t)\Big\{\sum_{\alpha,\beta=1}^{n}[a_{\alpha\beta}(A,t)-a_{\alpha\beta}(M,\Theta)]w^{M,\theta}_{x_{\alpha}x_{\beta}}(A,t;\,M,\Theta)+$$

$$+\sum_{\alpha=1}^{n}b_{\alpha}(A,t)w^{M,\theta}_{x_{\alpha}}(A,t;\,M,\Theta)+c(A,t)w^{M,\theta}(A,t;\,M,\Theta)\Big\}$$

*) V est une consequence de III.

où

$$\lambda(A,t)=\sqrt{\det|a^{\alpha\beta}(A,t)|}\,.$$

On trouve facilement, en posant $h_1=\min(h,2h')$ et $1-\frac{h_1}{2}<\mu_1<1$ que

$$(9)\qquad \int_\tau^t\int\!\!\int_{\Omega'}\!\!\int |N(A,t;\,M,\Theta)|\,dM\,d\Theta\leqslant$$

$$\leqslant C\left[\frac{1}{1-\mu_1}\int\!\!\int_{\Omega'}\!\!\int r_{AM}^{-n-2+h_1+2\mu_1}\,dM+\frac{2}{n-1}\int\!\!\int_{\Omega'}\!\!\int dM\right]\leqslant b,\ A\in\Omega'$$

et la constante C ne dépend que des coefficients de l'équation (i).

La constante b dépend donc seulement de ces coefficients et du domaine Ω.

Nous introduisons la fonction $\dot{N}(A,t;\,B,\Theta)$, qui est la dominante de la fonction $N(A;\,t;\,B;\,\Theta)$, de la manière suivante: dans la quasi-solution $w^{M,\theta}(A,t;\,B,\tau)$ et dans ses deux dérivées on remplace les coefficients par les bornes supérieures de leurs modules et la fonction $\exp[-\vartheta^{M,\theta}(A,B)/4(t-\tau)]$ par la fonction majorante $\exp.[-gr_{AB}^2/(t-\tau)]$; $g>0$; outre cela, les différences $a_{\alpha\beta}(A,t)-a_{\alpha\beta}(M,\Theta)$, sont remplacées par les bornes supérieures de leurs modules, qui sont données par l'inégalité (1″) d'Hölder; les autres coefficients de la fonction $N(A,t;\,M;\,\Theta)$, (8″) sont aussi remplacés par les bornes supérieures de leurs modules.

VI. Notre dernière supposition dont on parlait dans l'introduction est la suivante:

$$\int_\tau^t\int\!\!\int_{\Omega'}\!\!\int \dot{N}(A,t;B,\Theta)\,dB\,d\Theta\leqslant b\leqslant(2\sqrt{\pi})^n\omega\,,\qquad 0<\omega<1.$$

L'équation limite de l'introduction c'est l'équation elliptique normale suivante:

$$(10)\qquad \bar{\psi}(u)=\sum_{\alpha,\beta=1}^{n}\bar{a}_{\alpha\beta}(A)\,u''_{x_\alpha x_\beta}+\sum_{\alpha=1}^{n}\bar{b}_\alpha(A)\,u'_{x_\alpha}+\bar{c}(A)u=0\,,\qquad A\in\Omega\,.$$

On posera

$$\bar{\lambda}(A)=\lim_{t\to\infty}\lambda(A,t)=\sqrt{\det|\bar{a}^{\alpha\beta}(A)|}$$

et on désignera par $\bar{\omega}^B(A,B)$ la quasi-solution de l'équation (10):

$$\bar{w}^M(A,B)=2^{n-2}\Gamma\left(\frac{n}{2}-1\right)[\bar{\vartheta}^M(A,B)]^{-\frac{n}{2}+1},\qquad A,B,M,\in\Omega'\,.$$

Pareillement $\Gamma(A,Q)$ désignera la solution fondamentale de l'équation (10) et on aura:

$$\bar{\Gamma}(A,Q)=\bar{w}^Q(A,Q)+\int\!\!\int_{\Omega'}\!\!\int\bar{w}^B(A,B)\bar{\Phi}(B,Q)\,dB\,,\qquad A\in\Omega\,,$$

où pour:

$$\bar{f}(A, B) = \lambda \cdot \lambda(A)\overline{\varphi}_A[\overline{w}^B(A, B)], \qquad A, B \in \Omega',$$

on a

$$\bar{\Phi}(B, Q) = \bar{f}(B, Q) + \lambda \int\int\int_{\Omega'} \overline{\mathfrak{N}}(B, M)\bar{f}(M, Q)dM; \qquad \lambda = (2\sqrt{\pi})^{-n}$$

et $\overline{\mathfrak{N}}(B, M)$ est le noyau résolvant de l'équation (10). On y prouve, sous les hypothèses I-V, que la limite du noyau résolvant $\mathfrak{N}(B, t; M, t-\Theta)$ de l'équation de Volterra, déjà mentionnée, existe, pour $t \to \infty$, $0 < \Theta$ borné. Posons en effet:

$$\lim_{t\to\infty} \mathfrak{N}(B, t, M, t-\Theta) = \overline{\mathfrak{N}}(B, M, \Theta), \qquad 0 < \Theta \leqslant \text{const}.$$

On peut écrire, grâce à l'hypothèse VI, que:

$$\int_0^\infty \overline{\mathfrak{N}}(B, M, \Theta)d\Theta = \overline{\mathfrak{N}}(B, M).$$

On reçoit ainsi le résultat cherché sous la forme du

THÉORÈME 1. *Si les hypothèses* I—VI *sont verifiées, alors*

$$\lim_{t\to\infty} U(A, t) = \lim_{t\to\infty} \int_\tau^t \int\int_s \varphi(Q, \tau)\Gamma(A, t; Q, \tau)dQ\,d\tau =$$

$$= \int\int_s \overline{\varphi}(Q)\overline{\Gamma}(A, Q)dQ = \overline{U}(A),$$

c. à. d. la limite du potentiel $U(A, t)$ de simple couche de densité

$$\varphi(Q, \tau)$$

relat. à l'équation parabolique $\hat{\psi}(u) = 0$, (i) est égale au potentiel $\overline{U}(A)$ de densité limite $\varphi(Q) = \lim_{t\to\infty} \varphi(Q, t)$ relat. à l'équation limite $\bar{\psi}(u) = 0$, (10), qui est une équation elliptique normale.

La démonstration détaillée sera publiée dans Annali d. Sc. Normale Superiore di Pisa, 1958.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

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MATHEMATICS

On Absolute Convergence of Fourier Series of Almost All Functions of Wiener Space

by
Z. CIESIELSKI

Presented by W. ORLICZ on May 27, 1958

Denote by C the space of real functions $x(t)$ which are continuous on the interval $0 \leqslant t \leqslant 1$ and which vanish at $t = 0$. Using Wiener's method of integration [5] in the space C we establish the theorems given below. The measure in this space is built up in exactly the same way as the ordinary Lebesgue measure, except that the definition of an interval and its measure are different. The whole space has measure unity.

Denote by $\{\varphi_n\}$, $\{\chi_n\}$ and $\{\psi_n\}$ the following sets (complete and orthonormal in $L^2\langle 0,1\rangle$): trigonometrical, Haar's and Walsh's (see [2]). Let $\{\omega_n\}$ denote the set which coincides with one of these complete and orthonormal sets.

Let ω be a real function defined on $\langle 0, \infty)$ and satisfying the following conditions:

1° $\omega(0) = 0$,

2° $\omega(t)$ is increasing,

3° $\omega(2t)/\omega(t)$ is bounded as $t \to \infty$.

We shall say that the function ω satisfies condition (c) if the series

$$(*) \qquad \sum_{n=1}^{\infty} \frac{1}{\omega(n)}$$

is convergent, and if it satisfies the conditions 1°, 2° and 3°. If the series $(*)$ is divergent and the conditions 1°, 2° and 3° are satisfied, then we shall say that the function ω satisfies condition (d).

Further we use the following notations:

$$a_n[x] = \int_0^1 x(t)\,\omega_n(t)\,dt\,,$$

$$S_n^{(\alpha)}[x] = \sum_{k=1}^{n} |a_k[x]|^{\alpha}\,, \qquad S_n^{(1)}[x] = S_n[x]$$

and

$$R_n^{(\alpha)}[x] = \sum_{k=n+1}^{\infty} |a_k[x]|^\alpha .$$

THEOREM 1. *Let $0 < \alpha < 1$. If ω satisfies condition (c), then, for almost every $x \in C$ (except a set of Wiener's measure zero), there exists a positive integer $n[x]$ such that*

$$\sum_{n=n[x]}^{\infty} \frac{1}{n\omega\{\log S_n^{(\alpha)}[x]\}} < \infty . \tag{1}$$

THEOREM 2. *Let $0 < \alpha < 1$. If ω satisfies condition (d), then, for almost every $x \in C$, there is a positive integer $n[x]$ such that*

$$\sum_{n=n[x]}^{\infty} \frac{1}{n\omega\{\log S_n^{(\alpha)}[x]\}} = \infty . \tag{2}$$

THEOREM 3. *For almost all $x \in C$ there are positive integers $n[x]$ such that*

$$\sum_{n=n[x]}^{\infty} \frac{1}{n\omega\{S_n[x]\}} < \infty , \tag{3}$$

whenever the function ω satisfies condition (c).

THEOREM 4. *If ω satisfies condition (d), then there exists, for almost every $x \in C$, a positive integer $n[x]$ such that*

$$\sum_{n=n[x]}^{\infty} \frac{1}{n\omega\{S_n[x]\}} = \infty . \tag{4}$$

THEOREM 5. *Let ω satisfy condition (c) and let $\beta > 1$. Then, for almost every $x \in C$, there is a positive integer $n[x]$ such that*

$$\sum_{n=n[x]}^{\infty} \frac{1}{n\omega\{-\log R_n^{(\beta)}[x]\}} < \infty . \tag{5}$$

THEOREM 6. *Let $\beta > 1$. If ω satisfies condition (d), then, for almost every $x \in C$, there is a positive integer $n[x]$ such that*

$$\sum_{n=n[x]}^{\infty} \frac{1}{n\omega\{-\log R_n^{(\beta)}[x]\}} = \infty . \tag{6}$$

Let x_0 be an element of C such that

$$\lim_{n\to\infty} S_n^{(\alpha)}[x_0] = \infty$$

for all $\alpha < 2$. We shall say that x_0 has *Carleman's singularity*. For the set $\{\varphi_n\}$ an example of such function was given by Carleman [1], and for the sets $\{\chi_n\}$ and $\{\psi_n\}$ an existence of such function was given by Orlicz [3], [4]. Moreover, Orlicz found that the set of functions $x \in C$, having singularity of Carleman with respect to $\{\omega_n\}$, forms a residual set in C. On the other hand, by Theorem 5 the set of functions $x \in C$, having Carleman's singularity, forms a set of Wiener's measure zero.

Let us remark that it follows from Theorem 3 that the set of functions satisfying condition

$$\lim_{n\to\infty} S_n[x] < \infty$$

also forms a set of Wiener's measure zero.

The proofs of the theorems given in this note and their conclusions will be published in Studia Mathematica.

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MATHÉMATIQUE

Sur certains problèmes aux limites pour l'équation semi-linéaire parabolique normale

par
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Présenté par T. WAŻEWSKI le 4 juin, 1958

Cette communication contient la solution du problème aux limites mixtes avec la condition initiale homogène et non-homogène pour l'équation semi-linéaire parabolique normale de la forme:

$$(1)\qquad \begin{aligned} H[u(X,t)] &\equiv \sum_{ij=1}^{n} a_{ij}(X,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k(X,t)\frac{\partial u}{\partial x_k} + c(X,t)u - \frac{\partial u}{\partial t} \\ &= F\left(X, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right), \end{aligned}$$

où $X = (x_1, \ldots, x_n)$ désigne le point variable du domaine borné Ω dans l'espace euclidien $E^{(n)}$ à n-dimensions, limité par l'hypersurface fermée S, t — la variable de temps.

En général le problème consiste à trouver une fonction $u(X,t)$, qui satisferait:

1° à l'équation donnée (1) en tout point intérieur (X,t) du produit cartésien $\Omega \times (0,T)$ $(T>0)$

2° à la condition limite pour la dérivée transversale:

$$(2)\qquad \frac{du}{dv_P} = G[P,t,u(P,t)] \qquad (P,t) \in S \times (0,T]$$

3° à la condition initiale:

$$(3)\qquad \lim_{t\to 0} u(X,t) = f(X) \qquad X \in \Omega\,.$$

Les hypothèses admises sont les suivantes:

I. La surface S limitant le domaine Ω vérifie les conditions connues de Liapounoff [2].

II. Les coefficients $a_{ij}(X,t)$, $b_k(X,t)$ et $c(X,t)$ sont des fonctions:
a) déterminées et continues dans le produit cartésien:

$$(\Omega+S)\times[0,T]$$

b) vérifiant les conditions d'Hölder:

$$|a_{ij}(X,t)-a_{ij}(Y,\tau)|\leqslant K_1(r_{XY}^{\beta}+|t-\tau|^{\beta'}) \quad (i,j=1,\ldots,n)$$

$$|b_k(X,t)-b_k(Y,t)|\leqslant K_1 r_{XY}^{\beta} \quad (k=1,\ldots,n)$$

$$|c(X,t)-c(Y,t)|\leqslant K_1 r_{XY}^{\beta} \quad (\text{où } 0<\beta\leqslant 1\ ;\ 0<\beta'\leqslant 1)$$

dans le cas où $f(X)\equiv 0$ (condition initiale homogène):

1. III. La fonction $F(X,t,v_0,v_1,\ldots,v_n)$;
a) est déterminée et continue dans le produit cartésien:

$$(\Omega+S)\times[0,T]\times\{(v_0,v_1,\ldots,v_n)\,|\,|v_k|\leqslant R;\quad k=0,1,\ldots,n\}$$

b) vérifie la condition d'Hölder:

$$|F(X,t,v_0,v_1,\ldots,v_n)-F(Y,t,w_0,w_1,\ldots,w_n)|\leqslant K_2(r_{XY}^{\gamma}+\sum_{i=0}^{n}|v_i-w_i|^{\gamma'})$$

$$\text{où}\quad 0<\gamma\leqslant 1;\ 0<\gamma'\leqslant 1;$$

1. IV. La fonction $G(P,t,v)$:
a) est déterminée et continue dans le produit cartésien:

$$S\times[0,T]\times\{(v)\,|\,|v|\leqslant R\}\,,\quad \text{avec la propriété}\quad G(P,0,v)=0$$

b) vérifie la condition d'Hölder:

$$|G(P,t,v)-G(Q,t,w)|\leqslant K_3(r_{PQ}^{\delta}+|v-w|^{\delta'})\,,\quad \text{où}\quad 0<\delta\leqslant 1;\ 0<\delta'\leqslant 1;$$

et dans le cas où $f(X)\not\equiv 0$ (condition initiale non-homogène)

2. III. La fonction $F(X,t,v_0,v_1,\ldots,v_n)$:
a) est déterminée, bornée et continue dans le produit cartésien:

$$(\Omega+S)\times[0,T]\times\{(v_0,v_1,\ldots,v_n)\,|\,|v_k|<\infty;\ k=0,1,\ldots,n\};$$

b) vérifie la condition:

$$|F(X,t,v_0,v_1,\ldots,v_n)-F(Y,t,w_0,w_1,\ldots,w_n)|$$

$$\leqslant K_2\Big(r_{XY}^{\mathring{\gamma}}+t^{\varkappa}|v_0-w_0|^{\gamma}+t^{\varkappa'}\sum_{i=1}^{n}|v_i-w_i|^{\gamma'}\Big)\,,$$

où

$$0<\mathring{\gamma}\leqslant 1;$$

$$0<\gamma\leqslant 1;\quad \frac{1}{2}<\frac{\varkappa}{\gamma};\quad \frac{1}{2}<\frac{\varkappa'}{\gamma'};$$

$$0<\gamma'\leqslant 1;$$

2. IV. La fonction $G(P, t, v)$:

a) est déterminée, bornée et continue dans le produit cartésien:

$$S \times [0, T] \times \{(v) \mid |v| < \infty\}$$

b) vérifie la condition d'Hölder:

$$|G(P,t,v) - G(Q,t,w)| \leqslant K_3(r_{PQ}^{\delta} + |v-w|^{\delta'}) \quad \text{où} \quad 0 < \delta \leqslant 1;\ 0 < \delta' \leqslant 1\,.$$

2. V. La fonction $f(X)$ est déterminée et continue dans le domaine Ω.

En s'appuyant sur les propriétés du potentiel généralisé de charge spatiale, potentiel généralisé de simple couche et intégrale de Poisson-Weierstrass généralisée relatifs à l'équation:

$$H[u(X,t)] = 0 \tag{4}$$

(voir [1], [2], [3]) on amène le problème mixte à la résolution du système des deux équations intégro-différentielles à deux fonctions inconnues $u(X, t)$ et $\varphi(P, t)$. Ensuite on est conduit au système des $n+2$ équations intégrales à $n+2$ fonctions inconnues $u_0(X, t)\ u_1(X, t), \ldots, u_n(X, t), \varphi(P, t)$ de la forme:

$$\left\{\begin{aligned}
&u_k(X,t) = \\
&= \int_0^t \iiint_\Omega \Gamma_k(X,t;\,Y,\tau)\widetilde{F}[Y,\tau,u_0(Y,\tau),u_1(Y,\tau),\ldots,u_n(Y,\tau)]\,dY\,d\tau + \\
&+ \int_0^t \iint_S \Gamma_k(X,t;Q,\tau)\varphi(Q,\tau)\,dQ\,d\tau + \iiint_\Omega \Gamma_k(X,t;\,Y,0)\widetilde{f}(Y)\,dY; \\
&\qquad\qquad (k = 0, 1, \ldots, n) \\
&\varphi(P,t) - \int_0^t \iint_S \lambda(P,t)\frac{d\Gamma(P,t;\,Q,\tau)}{dv_P}\varphi(Q,\tau)\,dQ\,d\tau \\
&= \lambda(P,t)\Big\{\int_0^t \iiint_\Omega \frac{d\Gamma(P,t;\,Y,\tau)}{dv_P}\widetilde{F}[Y,\tau,u_0(Y,\tau),\ldots,u_n(Y,\tau)]\,dY\,d\tau + \\
&- G\Big[P,t,\int_0^t \iiint_\Omega \Gamma_0(P,t;Y,\tau)\times \\
&\times \widetilde{F}[Y,\tau,u_0(Y,\tau),u_1(Y,\tau),\ldots,u_n(Y,\tau)]\,dY\,d\tau - \\
&+ \iiint_\Omega \Gamma_0(P,t;\,Y,0)\widetilde{f}(Y)\,dY + \int_0^t \iint_S \Gamma_0(P,t;Q,\tau)\varphi(Q,\tau)\,dQ\,d\tau\Big] + \\
&+ \iiint_\Omega \frac{d\Gamma(P,t;\,Y,0)}{dv_P}\widetilde{f}(Y)\,dY\Big\}
\end{aligned}\right. \tag{5}$$

où

$$\lambda(P,t) \overset{\text{df}}{=} \frac{2(\det[a^{ij}(P,t)])^{1/2}}{(2\sqrt{\pi})^n};$$

$$\Gamma_k(X,t;\ Y,\tau) \overset{\text{df}}{=} \begin{cases} = \Gamma(X\,t;Y,\tau); & k=0 \\ = \dfrac{\partial\Gamma(X,t;\ Y,\tau)}{\partial x_k}; & k=1,\ldots,n \end{cases}$$

$$\widetilde{F}(X,t,v_0,v_1,\ldots,v_n) \overset{\text{df}}{=} -\frac{\lambda(X,t)}{2} F(X,t,v_0,v_1,\ldots,v_n);$$

$$\widetilde{f}(X) \overset{\text{df}}{=} \frac{\lambda(X,0)}{2} f(X);$$

$\Gamma(X,t;\ Y,\tau)$ — solution fondamentale de l'équation (4) [1].

Dans le premier cas où $f(X) = 0$, on exprime le système (5) de la façon suivante:

$$(\widetilde{5}) \qquad (u_0,u_1,\ldots,u_n,\varphi) = \widetilde{H}(u_0,u_1,\ldots,u_n,\varphi),$$

où $\widetilde{H}$ désigne une opération fonctionnelle définie par les membres droits des équations (5).

On a résolu ce système à l'aide du théorème topologique de J. Schauder [4] sur le point invariant d'une opération fonctionnelle. Dans ce but on considère un espace linéaire, normé et complet $\mathfrak{X}$, composé de tous les systèmes des fonctions

$$W = [w_0(X,t), w_1(X,t), \ldots, w_n(X,t), \varrho(P,t)]$$

définies et continues dans les produits cartésiens $(\Omega + S)\times[0,T]$, resp. $S\times[0,T]$, la norme du point W de cet espace $\mathfrak{X}$, est définie par la formule:

$$\|W\| \overset{\text{df}}{=} \sum_{k=0}^{n} \sup_{(X,t)} |w_k(X,t)| + \sup_{(P,t)} |\varrho(P,t)|.$$

Dans cet espace $\mathfrak{X}$ on considère un ensemble ferme et convexe E de points W défini par les relations suivantes:

$$(6) \quad E \overset{\text{df}}{=} \{W = (w_0,w_1,\ldots,w_n,\varrho) \,|\, |w_k(X,t)| \leqslant R; \quad k=0,1,\ldots,n;$$

$$|\varrho(P,t)| \leqslant R_1; \quad \varrho(P,0) = 0; \quad |\varrho(P,t) - \varrho(Q,t)| \leqslant K r_{PQ}^n;\}\,;$$

En tenant compte du système $(\widetilde{5})$ on transforme l'ensemble E à l'aide des relations:

$$(7) \qquad (z_0,z_1,\ldots,z_n,\zeta) = \widetilde{H}(w_0,w_1,\ldots,w_n,\varrho)$$

qui font correspondre à tout point $(w_0,w_1,\ldots,w_n,\varrho)$ de l'ensemble E un point $(z_0,z_1,\ldots,z_n,\zeta)$ de l'espace $\mathfrak{X}$. On peut montrer que l'ensemble

transforme $\widetilde{H}(E)$ fera partie de l'ensemble E, si les inégalités suivantes sont vérifiées:

$$C_0|F| + C_1|G| \leqslant R_1 \leqslant \frac{R}{C_2} - C_3|F| \tag{8}$$

$$C_4|F| + C_5|G| + k_3(C_6|F| + C_7 R_1)^{\delta'} \leqslant K \leqslant \frac{R}{C_8} - C_9(|F| + C_{10} R_1)\,, \tag{9}$$

où $C_0, C_1, \ldots, C_{10}$ – constantes positives elles aussi ne dépendent que de la surface S du domaine Ω, de la constante T et des coefficients de l'équation (1).

$$|F| \overset{\mathrm{df}}{=} \sup_{(\ldots)} |F(\ldots)|\,.$$

On démontre ensuite que l'opération (7) est continue dans l'espace $\mathfrak{X}$ et que l'ensemble transforme $\widetilde{H}(E)$ est compact.

On en conclut, d'après le théorème de Schauder, l'existence de la solution du problème, exprimée dans le théorème suivant:

THÉORÈME I. *Si les coefficients de l'équation parabolique* (1), *les fonctions données* $F(X, t, v_0, v_1, \ldots, v_n)$, $G(P, t, v)$ *et la surface fermée* S *limitant le domaine borné* Ω, *vérifient les conditions* II., 1. III., 1. IV. *et* I., *si en outre les inégalités* (8), (9) *sont satisfaites, alors il existe au moins une fonction* $u(X, t)$, *qui vérifie l'équation donnée* (1) *en tout point intérieur* (X, t) *du produit cartésien* $(\Omega + S) \times [0, T]$, *qui vérifie la condition limite* (2) *pour* $(P, t) \in S \times [0, T]$ *et qui vérifie la condition initiale* (3) *homogène* $(f(X) \equiv 0)$ *en tout point intérieur* X *du domaine* Ω.

Dans le second cas, où $f(X) \not\equiv 0$, on peut exprimer le système (5) de la façon suivante:

$$(u_0, u_1, \ldots, u_n, \varphi) = \underset{\sim}{H}(u_0, u_1, \ldots, u_n, \varphi)\,, \tag{5}$$

où $\underset{\sim}{H}$ désigne une opération fonctionnelle définie par les membres droits des équations (5).

La preuve d'existence de la solution dans ce cas subit une modification, parce que les fonctions inconnues

$$u_0(X, t), u_1(X, t), \ldots, u_n(X, t) \quad \text{et} \quad \varphi(P, t)\,,$$

vérifiant le système (5), sont non-bornées si $t \to 0$.

Pour éviter cette singularité on multiplie les fonctions $u_k(X, t)$ $(k = 0, 1, \ldots, n)$ et $\varphi(P, t)$ par t^{χ}, où

$$\chi = \begin{cases} = \min\left(\dfrac{\varkappa}{\gamma}, \dfrac{\varkappa'}{\gamma'}\right); & \text{pour } \dfrac{\varkappa}{\gamma} < 1\,, \quad \text{où } \dfrac{\varkappa'}{\gamma'} < 1; \\ = \Delta; & \text{où } \dfrac{1}{2} < \Delta < 1; \quad \text{pour } \dfrac{\varkappa}{\gamma} \geqslant 1\,, \quad \text{et } \dfrac{\varkappa'}{\gamma'} \geqslant 1; \end{cases}$$

En conséquence on arrive à l'opération fonctionnelle suivante:

$$(10) \qquad (z_0, z_1, \ldots, z_n, \zeta) = t^{\varkappa} H\left(\frac{w_0}{\tau^{\varkappa}}, \frac{w_1}{\tau^{\varkappa}}, \ldots, \frac{w_n}{\tau^{\varkappa}}, \frac{\varrho}{\tau^{\varkappa}}\right),$$

pour laquelle la preuve d'existence du point invariant est analogique comme pour le problème dans le premier cas.

On peut alors énoncer le théorème suivant:

THÉORÈME II. *Si les coefficients de l'équation parabolique* (1), *les fonctions données* $F(X, t, v_0, v_1, \ldots, v_n)$, $G(P, t, v)$, $f(X)$ *et la surface fermée* S *limitant le domaine borné* Ω *vérifient les conditions* II., 2. III., 2. IV., 2. V. *et* I., *alors il existe au moins une fonction* $u(X, t)$, *qui vérifie l'équation donnée* (1) *en tout point intérieur* (X, t) *du produit cartésien* $(\Omega + S) \times [0, T]$, *qui vérifie la condition limite* (2) *pour* $(P, t) \in S \times (0, T]$, *et qui vérifie la condition initiale* (3) *non-homogène* $(f(X) \not\equiv 0)$ *en tout point intérieur* X *du domaine* Ω.

Les démonstrations plus détaillées seront publiées dans le Bulletin de l'Académie Technique Militaire à Varsovie.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

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MATHÉMATIQUE

Sur les involutions multivalentes

par

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Présenté par K. KURATOWSKI le 6 juin 1958

Cette communication contient une généralisation du lemme de Kuratowski concernant les involutions continues définies sur les continus localement connexes et unicohérents (voir [1], p. 90 et [3], p. 117). Il va être montré comment le théorème général sur les fonctions multivalentes semicontinues supérieurement (voir le théorème 1) entraîne le lemme de Kuratowski généralisé aux involutions multivalentes dites ici au sens large (voir le théorème 3), donc aussi au sens étroit (voir le théorème 4), qui embrasse en particulier les involutions ordinaires (c'est-à-dire univalentes).

1. Fonctions multivalentes. X étant un espace compact, on appelle une fonction $F\colon X \to 2^X$ *semicontinue supérieurement* (voir [2], p. 32) lorsque $\lim\limits_{n\to\infty} x_n = x$ entraîne $\operatorname{Ls}\limits_{n\to\infty} F(x_n) \subset F(x)$. Posons pour tout $A \subset X$

$$F(A) = \bigcup_{x \in A} F(x) .$$

Il est facile de voir que la semicontinuité supérieure de la fonction F équivaut à la condition

$$\operatorname{Ls}_{i\to\infty} F(A_i) \subset F(\operatorname{Ls}_{i\to\infty} A_i) .$$

Soit (α) la propriété suivante de l'ensemble A:

(α) *Il existe une composante C de l'ensemble $X - A$, telle que $F(A) \subset C$.*

Il est aisé de constater que cette propriété passe aux sous-ensembles: si $B \subset A$ et A a la propriété (α), B l'a également.

THÉORÈME 1. *L'espace X étant un continu localement connexe et unicohérent, si la fonction $F\colon X \to 2^X$ est sémicontinue supérieurement et l'ensemble $G \subset X$ est ouvert, alors G ou bien $X - G$ a une composante dépourvue de la propriété* (α).

Démonstration. Par suite de la compacité de X, l'ensemble G (supposé évidemment non-vide) est formé tout au plus d'une infinité *dénombrable* de composantes G_i:

$$G = \bigcup_{i=1}^{\nu} G_i \quad \text{où} \quad \nu \leqslant \omega , \tag{1}$$

et elles sont des régions par suite de la connexité locale de X.

Admettons que tout G_i a la propriété (α). Il existe donc autant de composantes C_i de $X - G_i$, telles que

$$F(G_i) \subset C_i \quad \text{pour} \quad i = 1, 2, \ldots, \nu , \tag{2}$$

et elles sont des continus. On a $\operatorname{Fr}(C_i) \neq 0$ pour chacun de ces i par suite de la connexité de X et de la conséquence $0 \neq C_i \neq X$ de (2), G étant non-vide. Soit $x \in \operatorname{Fr}(C_i)$. Par conséquent, $x = \lim_{n\to\infty} x_n$ où $x_n \in G_i$ pour tout $n = 1, 2, \ldots$, d'où $\operatorname{Ls}_{n\to\infty} F(x_n) \subset F(x)$ par suite de la semicontinuité supérieure de la fonction F. On a en même temps $\operatorname{Ls}_{n\to\infty} F(x_n) \subset C_i$ d'après (2) et parce que tout C_i est fermé; on a enfin $\operatorname{Ls}_{n\to\infty} F(x_n) \neq 0$ par suite de la compacité de X. Ainsi, $C_i \cap F(x) \neq 0$, donc

$$C_i \cap F[\operatorname{Fr}(C_i)] \neq 0 \quad \text{pour} \quad i = 1, 2, \ldots, \nu . \tag{3}$$

Soit $\{G_{i_j}\}$, où $j = 1, 2, \ldots, \mu \leqslant \nu$, une suite partielle de la suite $\{G_i\}$, définie comme il suit:

Posons $i_1 = 1$ et admettons que, $i_1, \ldots, i_k$ étant déjà définis, on a

$$i_j < i_{j+1} \quad \text{pour} \quad j = 1, 2, \ldots, k-1 . \tag{4}$$

Les composantes étant disjointes, on a en particulier $G_j \cap G_m = 0$ pour $j \leqslant i_k < m$, c'est-à-dire $G_m \subset X - G_j$. Par suite de la connexité de G_m, il existe donc une seule composante de $X - G_j$ qui contient G_m. On a donc soit $G_m \subset C_j$, soit $G_m \cap C_j = 0$, en particulier pour $j \leqslant i_k < m$, d'où soit $G_m \subset \bigcap_{j=1}^{k} C_{i_j}$, soit $G_m \cap \bigcap_{j=1}^{k} C_{i_j} = 0$ en vertu de (4). Soit i_{k+1} le plus petit des indices $m > i_k$ pour lesquels $G_m \subset \bigcap_{j=1}^{k} C_{i_j}$; s'il n'y en a pas, soit $k = \mu$. On a par conséquent $i_k < i_{k+1}$, c'est-à-dire (4) pour $j = 1, 2, \ldots, k$.

La suite ascendante $\{i_j\}$ où $j \leqslant \mu$, et par conséquent la suite $\{G_{i_j}\}$, se trouvent ainsi définies de façon que

$$G_{i_k} \subset \bigcap_{j-1}^{k-1} C_{i_j} \quad \text{pour} \quad k = 2, 3, \ldots, \mu . \tag{5}$$

Posons pour tout $k = 1, 2, \ldots, \mu$

$$H_k = X - C_{i_k} \quad \text{et} \quad L_k = \bigcap_{j=1}^{k} C_{i_j} . \tag{6}$$

Les ensembles H_k sont donc des régions (voir [2], p. 88) et les ensembles L_k sont compacts. En outre, en vertu de (5), $j < n$ entraîne $G_{i_n} \subset C_{i_j}$, c'est-à-dire $H_j \subset X - G_{i_n}$; par suite de la connexité de H_k, il existe donc une seule composante de $X - G_{i_n}$ qui contient H_k. On a donc en particulier soit $H_j \subset C_{i_n}$, soit $H_j \cap C_{i_n} = 0$, d'où en vertu de (6)

(7) soit $H_j \cap H_n = 0$, soit $H_j \subset H_n$ pour tout $j < n$.

Remarquons que

(8) $$G_m \cap \bigcap_{j=1}^{\mu} C_{i_j} = 0 \quad \text{pour tout} \quad m = 1, 2, \ldots, \nu .$$

En effet, si $m = i_k$ pour un $k = 1, 2, \ldots, \mu$, on a $C_{i_k} \subset X - G_m$ par définition, d'où à plus forte raison $G_m \cap \bigcap_{j=1}^{\mu} C_{i_j} = 0$. Si, par contre, $m \neq i_k$ où $k = 1, 2, \ldots, \mu$, on a $G_m \cap \bigcap_{j=1}^{h} C_{i_j} = 0$ (d'après la définition de la suite $\{G_{i_j}\}$) où $i_h = \max_{i_k < m} i_k$ (donc $h \leqslant \mu$).

En vertu de (6) et (8), on a $(\bigcup_{m=1}^{\nu} G_m) \cap \bigcap_{k=1}^{\mu} L_k = 0$, c'est-à-dire

(9) $$\bigcap_{k=1}^{\mu} L_k \subset X - G$$

en vertu de (1). On a $\mathrm{Fr}(C_{i_k}) \subset \mathrm{Fr}(X - G_{i_k})$ (voir [2], p. 169, 3), où $\mathrm{Fr}(C_{i_k}) = \mathrm{Fr}(X - C_{i_k}) = \mathrm{Fr}(H_k)$ d'après (6) et $\mathrm{Fr}(X - G_{i_k}) = \mathrm{Fr}(G_{i_k}) \subset \subset L_{k-1}$, car $G_{i_k} \subset L_{k-1}$ d'après (5) et (6) et L_{k-1} est compact. Par conséquent,

(10) $$\mathrm{Fr}(H_k) \subset L_{k-1} \quad \text{pour} \quad k = 2, 3, \ldots, \mu .$$

Posons

(11) $$L = \bigcap_{k=1}^{\mu} C_{i_k}, \quad \text{c'est-à-dire} \quad L = \bigcap_{k=1}^{\mu} L_k$$

d'après (6). On a $X - L = \bigcup_{k=1}^{\mu} (X - C_{i_k}) = \bigcup_{k=1}^{\mu} H_k$ en vertu de (11) et (6). Toute composante de $X - L$ est donc, en vertu de (7), somme d'une suite non-descendante de régions qui est une suite partielle de $\{H_k\}$. Ainsi en supposant que L ait la propriété (a), donc qu'il existe une composante D de $X - L$ telle que

(12) $$F(L) \subset D ,$$

on aurait en particulier

(13) $$D = \bigcup_{m=1}^{\lambda} H_{k_m}, \quad \text{où} \quad \lambda \leqslant k_\lambda \leqslant \mu \quad \text{et} \quad H_{k_m} \subset H_{k_{m+1}} \quad \text{pour} \quad m = 1, 2, \ldots, \lambda$$

et

$$D \cap H_j = 0 \quad \text{pour} \quad j > k_\lambda . \tag{14}$$

Deux cas peuvent se présenter:

1° $k_\lambda < \omega$. On a alors, pour $k_\lambda < j \leqslant \mu$, $H_{k_\lambda} \cap H_j = 0$ d'après (14), donc $H_{k_\lambda} \subset X - H_j = C_{i_j}$ d'après (6), d'où $\operatorname{Fr}(H_{k_\lambda}) \subset C_{i_j}$ (vu la compacité de C_{i_j}). On a en même temps $\operatorname{Fr}(H_{k_\lambda}) \subset L_{k_\lambda - 1}$ d'après (10) et $\operatorname{Fr}(H_{k_\lambda}) \subset C_{i_{k_\lambda}}$ d'après (6). Par conséquent $\operatorname{Fr}(H_{k_\lambda}) \subset L_{k_\lambda - 1} \cap C_{i_{k_\lambda}} \cap \bigcap_{k_\lambda < j \leqslant \mu} C_{i_j} = L_{k_\lambda} \cap$ $\cap \bigcap_{k_\lambda < j \leqslant \mu} C_{i_j} = L$ en vertu de (6) et (11). En remplaçant donc $\operatorname{Fr}(H_{k_\lambda})$ par L et H_{k_λ} par D (puisque $D = H_{k_\lambda}$ dans le cas 1°) dans la conséquence $F[\operatorname{Fr}(H_{k_\lambda})] \cap (X - H_{k_\lambda}) \neq 0$ de (3) et (6), il vient $F(L) \cap (X - D) \neq 0$, contrairement à (12).

2° $k_\lambda = \omega$. On a d'une part $\operatorname*{Ls}_{m \to \omega} \operatorname{Fr}(H_{k_m}) \subset \operatorname*{Ls}_{k \to \infty} \operatorname{Fr}(H_k) \subset \operatorname*{Ls}_{k \to \infty} L_{k-1} =$ $= \operatorname*{Lim}_{k \to \infty} L_k = L$ en vertu de (10) et (11) (puisque $\infty = \omega = k_\lambda = \lambda = \mu = \nu$ dans le cas 2°) et d'autre part $(X - H_{k_m}) \cap F[\operatorname{Fr}(H_{k_m})] \neq 0$ en vertu de (3) et (6), d'où $0 \neq \operatorname*{Ls}_{m \to \infty} (X - H_{k_m}) \cap \operatorname*{Ls}_{m \to \infty} F[\operatorname{Fr}(H_{k_m})] \subset \operatorname*{Ls}_{m \to \infty} (X - H_{k_m}) \cap$ $\cap F[\operatorname*{Ls}_{m \to \infty} \operatorname{Fr}(H_{k_m})]$, la fonction F étant semicontinue supérieurement par hypothèse. Ainsi $F(L) \cap \operatorname*{Ls}_{m \to \infty} (X - H_{k_m}) \neq 0$, d'où — contrairement à (12) — $F(L) \cap (X - D) \neq 0$, puisque $\operatorname*{Ls}_{m \to \infty} (X - H_{k_m}) = \bigcap_{m=1}^{\infty} (X - H_{k_m}) = X - D$ en vertu de (13). Il est donc démontré que L n'a pas de propriété (α).

Pour montrer que L est un continu, notons que $L_1 = C_{i_1} = C_1$ l'est d'après (6) et qu'en l'admettant pour L_{k-1}, L_k l'est également. En effet, H_k étant une région, (10) entraîne que $H_k \cup L_{k-1}$ est connexe et que l'on a $\overline{H_k \cup L_{k-1}} = H_k \cup \operatorname{Fr}(H_k) \cup L_{k-1} \subset H_k \cup L_{k-1}$. Ainsi $H_k \cup L_{k-1}$ est un continu. On a d'après (6) la décomposition suivante de l'espace X, qui est unicohérent par hypothèse:

$$X = [(X - C_{i_k}) \cup L_{k-1}] \cup C_{i_k} = (H_k \cup L_{k-1}) \cup C_{i_k} .$$

Par conséquent $(H_k \cup L_{k-1}) \cap C_{i_k}$ est aussi un continu et comme il est égal à $L_{k-1} \cap C_{i_k} = L_k$ d'après (6), il en est de même de L_k. La suite $\{L_k\}$ étant descendante d'après (6), L est un continu en vertu de (11).

Comme tel, L est contenu dans une seule composante de $X - G$, étant donné que l'on a $L \subset X - G$ en vertu de (9) et (11). L étant constaté dépourvu de la propriété (α), cette composante de $X - G$ l'est aussi, vu que la propriété (α) passe aux sous-ensembles.

2. Ensembles partiellement invariants. Un point x est dit *point invariant* de la fonction $F\colon X \to 2^X$ lorsque $x \in F(x)$. Plus généralement: appelons un ensemble $A \subset X$ *ensemble partiellement invariant* de cette

fonction lorsque $A \cap F(A) \neq 0$. Ainsi, dans le cas particulier où l'ensemble composé d'un seul point $\{x\}$ est un ensemble partiellement invariant de la fonction F, le point x est un point invariant de cette fonction.

$C(X) \subset 2^X$ désignant l'ensemble des sous-continus non-vides de X, on a le lemme évident qui suit:

LEMME 1. *Si la fonction* $F\colon X \to C(X)$ *est semicontinue supérieurement et l'ensemble* $A \subset X$ *est connexe, l'ensemble* $F(A)$ *l'est également.*

THÉORÈME 2. *L'espace* X *étant un continu localement connexe et unicohérent, si la fonction* $F\colon X \to C(X)$ *est semicontinue supérieurement et l'ensemble* $G \subset X$ *est ouvert, alors* G *ou bien* $X - G$ *a une composante partiellement invariante de la fonction* F.

Démonstration. D'après le théorème 1, il existe une composante A de G ou de $X - G$ privée de la propriété (α), c'est-à-dire telle que $F(A)$ n'est contenu dans aucune composante de $X - A$. En vertu du lemme 1, il est donc impossible que l'on ait $F(A) \subset X - A$. On a par conséquent $A \cap F(A) \neq 0$.

COROLLAIRE. *Sous les mêmes hypothèses, si l'on a en outre* $F(G) \subset G$ *et* $F(X - G) \subset X - G$, *alors* G *ou bien* $X - G$ *a une composante* A *telle que* $F(A) \subset A$.

3. Involutions multivalentes. Appelons une fonction $\Phi\colon X \to 2^X$ semicontinue supérieurement *involution multivalente*, ou *involution* tout court, lorsque $x_1 \in \Phi(x_2)$ pour tout $x_2 \in \Phi(x_1)$; en d'autres termes, lorsque $x_1 \in \Phi(x_2)$ équivaut à $x_2 \in \Phi(x_1)$ pour tout couple x_1, x_2 de points de X.

Appelons une fonction $\Phi\colon X \to 2^X$ semicontinue supérieurement *involution* (multivalente) *au sens large* lorsque $x_1 \in \Phi(x_2)$ pour au moins un $x_2 \in \Phi(x_1)$; en d'autres termes, lorsqu'on a $x \in \Phi(\Phi(x))$ pour tout point x de X. Évidemment, toute involution (tout court) l'est aussi au sens large.

REMARQUE 1. En convenant de dire qu'une fonction $F_1\colon X \to 2^X$ *renferme* la fonction $F_2\colon X \to 2^X$ lorsque $F_2(x) \subset F_1(x)$ pour tout $x \in X$, *toute involution au sens large* $\Psi\colon X \to 2^X$ *renferme une involution tout court* $\Phi\colon X \to 2^X$.

Soit en effet pour tout $x \in X$

$$\Phi(x) = \underset{x' \in \Psi(x)}{E} \{x \in \Psi(x')\}\,. \tag{15}$$

Évidemment, Ψ renferme Φ et d'après (15)

$$x' \in \Phi(x) \text{ équivaut à } x' \in \Psi(x) \text{ et } x \in \Psi(x'); \tag{16}$$

par conséquent $x' \in \Phi(x)$ entraîne $x \in \Phi(x')$. Reste à établir la semicontinuité supérieure de la fonction Φ. Soient $x = \lim\limits_{n\to\infty} x_n$ et $x' \in \operatorname{Ls}\limits_{n\to\infty} \Phi(x_n)$, c'est-à-dire $x' = \lim\limits_{i\to\infty} x'_{n_i}$ où $x'_{n_i} \in \Phi(x_{n_i})$. On a par conséquent $x_{n_i} \in \Psi(x'_{n_i})$ d'après (15), d'où $\lim\limits_{i\to\infty} x_{n_i} \in \operatorname{Ls}\limits_{i\to\infty} \Psi(x'_{n_i})$, donc $x \in \Psi(x')$, la fonction Ψ étant semicontinue

supérieurement. Pour la même raison, $x' \in \operatorname{Ls}_{n\to\infty} \Phi(x_n) \subset \operatorname{Ls}_{n\to\infty} \Psi(x_n) \subset \Psi(x)$, d'où $x' \in \Phi(x)$ en vertu de (16). Il est donc établi que $\operatorname{Ls}_{n\to\infty} \Phi(x_n) \subset \Phi(x)$.

LEMME 2. *Si la fonction* $\Psi\colon X \to 2^X$ *est une involution au sens large,* $\Psi(A) \subset A$ *entraîne* $\Psi(A) = A$.

En effet, $x \in A$ entraîne alors $x \in \Psi(\Psi(x))$ et $\Psi(x) \subset A$, d'où $x \in \Psi(A)$.

THÉORÈME 3. *L'espace* X *étant un continu localement connexe et unicohérent, si la fonction* $\Psi\colon X \to C(X)$ *est une involution au sens large et l'ensemble ouvert* $G \subset X$ *satisfait aux conditions* $\Psi(G) \subset G$ *et* $\Psi(X-G) \subset X-G$, *alors* G *ou bien* $X-G$ *a une composante* A *telle que* $\Psi(A) = A$.

C'est une conséquence directe du corollaire et du lemme 2.

REMARQUE 2. L'exemple suivant montre que le théorème 3 est en défaut si Ψ n'est pas une involution au sens large, même lorsque l'ensemble ouvert $G \subset X$ satisfait aux égalités $\Psi(G) = G$ et $\Psi(X-G) = X-G$, et même lorsque la fonction Ψ est univalente.

Soient X le carré aux sommets opposés $(0, -2)$ et $(4, 2)$; en désignant par $\langle p, q\rangle$ le segment rectiligne aux bouts p et q, posons:

$$L_1 = \langle (0,-2), (0,2)\rangle\,, \qquad M_i = \langle (2/2^i, -2), (2/2^i, -1)\rangle\,,$$
$$L_2 = \langle (4,-2), (4,2)\rangle\,, \qquad M^i = \langle (2/2^i, 1), (2/2^i, 2)\rangle\,,$$
$$L_3 = \langle (0,0), (4,0)\rangle\,, \qquad \text{pour } i = 0, 1, 2, \ldots,$$

$$X-G = \bigcup_{i=1}^{3} L_i \cup \bigcup_{i=0}^{\infty} (M_i \cup M^i)$$

et pour $(x, y) \in X$

$$f(x,y) = \begin{cases} (x, x-5) & \text{pour} \quad y \leqslant x-5\,, \\ (x,y) & \text{pour} \quad x-5 \leqslant y \leqslant 5-x\,, \\ (x, 5-x) & \text{pour} \quad 5-x \leqslant y\,, \end{cases}$$

$$g(x) = \begin{cases} 4x & \text{pour} \quad 0 \leqslant x \leqslant 1\,, \\ 6-2x & \text{pour} \quad 1 \leqslant x \leqslant 2\,, \\ x & \text{pour} \quad 2 \leqslant x \leqslant 4\,, \end{cases}$$

$$g_1(x,y) = (g(x), y)\,, \qquad g_2 = g_1 | f(X)\,,$$
$$h(x,y) = (x, -y)\,, \qquad \Psi = h g_2 f\,.$$

Ainsi définie, la fonction $\Psi\colon X \to X$ est continue univalente et on a $\Psi(G) = G$, de même que $\Psi(X-G) = X-G$ sans qu'il existe une composante A de G, pas plus que de $X-G$, telle que l'on ait $\Psi(A) = A$.

LEMME 3. *Si la fonction* $\Phi\colon X \to 2^X$ *est une involution (tout court), la condition* $\Phi(A) \subset A$ *entraîne la condition* $\Phi(X-A) \subset X-A$.

En effet, en supposant que l'on ait $\Phi(x') \cap A \neq 0$ pour un $x' \in X-A$, on aurait $x' \in \Phi(x)$ pour tout $x \in \Phi(x') \cap A$, c'est-à-dire $\Phi(x) \cap (X-A) \neq 0$, contrairement à l'hypothèse que $\Phi(A) \subset A$.

THÉORÈME 4. *L'espace X étant un continu localement connexe et unicohérent, si la fonction Φ: $X \to C(X)$ est une involution et l'ensemble ouvert $G \subset X$ satisfait à la condition $\Phi(G) \subset G$, alors G ou bien $X - G$ a une composante A telle que $\Phi(A) = A$.*

C'est une conséquence directe du théorème 3 et du lemme 3.

Pour les involutions au sens usuel de ce terme, c'est-à-dire univalentes, le théorème 4 coïncide avec le lemme de Kuratowski, cité au début.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

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Spin Waves Frequency Spectrum for Various Orders of Ferrimagnetic Translation Lattices

by
H. COFTA

Presented by W. RUBINOWICZ on May 21, 1958

Introduction

The purpose of this paper is the derivation of a general formula for the spin waves frequency spectrum and investigation of the behaviour of spin waves in ferrimagnetic translation lattices. The main question is the consideration of various possible orders of the magnetic superstructure. We also take into account magnetic anisotropy and interactions between more distant neighbours, in order to compare with the experiment the results obtained, as applied to several ferri- and antiferromagnetic lattices.

Classification of orders

In ferrimagnetics with two kinds of spins, the most simple order is the one in which any spin of the first kind is surrounded by spins of the second kind only, and *vice versa*. Let us call this order a "natural order". In the more complicated kinds of order not all nearest neighbours of a given spin belong to the other sublattice. We shall distinguish a class of orders, in which both the neighbours in sites $\vec{r}$ and $-\vec{r}$ relative to an arbitrary spin belong to the same sublattice.

Let us call it a "regular order". In ferrimagnetics with three kinds of spins the natural order cannot be realized, but other regular orders may exist. We shall also assume that the exchange integrals between the given spin and the spins in symmetric sites $\vec{r}$ and $-\vec{r}$ have the same value: $J(\vec{r}) = J(-\vec{r})$.

The method used

We use the semiclassical theory of spin waves. It represents a most convenient method of deriving a rigorous dispersion law, as well as of discussing the propagation of spin waves in non-regular orders. This

theory leads to a simple picture of a spin wave; it is a wave of spins precessing with angular velocity ω.

There are two ways of treating spin waves, based on the semiclassical theory. The first way treats formally the equation of motion of a given spin. In some special cases of ferrimagnetics it is used in [3] for ferrites, in [5] for simple cubic lattice, and in [6] for natural orders. The second way is the well-known kinematical approach, described in [4], here generalized to apply to any translation lattice. It consists in calculating ω from the precession equation $\dot{\vec{S}} = \vec{\omega} \times \vec{S}$, taking $\dot{\vec{S}}$ from the equation of motion.

General Dispersion Law

Derivation

We consider the case of three geometrically identical sublattices of a translation lattice, where each sublattice is formed of one of the different kinds of spins: $\vec{S}_{(1)}, \vec{S}_{(2)}, \vec{S}_{(3)}$. Then, the equation of spin motion for the i-th sublattice is given by

$$\dot{\vec{S}}_{(i)}(\vec{m}_i) = \vec{S}_{(i)}(\vec{m}_i) \times \left\{ \vec{P}_{(i)} + \frac{2}{\hbar^2} \sum_{j}^{3} \sum_{\vec{n}_{ij}} J_{\vec{m},\vec{n}_{ij}} \vec{S}_{(j)}(\vec{n}_{ij}) \right\}, \tag{1}$$

$J_{\vec{m},\vec{n}}$ being the exchange integral between two spins in sites $\vec{m}$ and $\vec{n}$ of the lattice. Besides,

$$\vec{P}_{(i)} = \gamma \mu_0 (\vec{H}_0 + \vec{H}_{(i)}) ,$$

where $\gamma = \frac{e}{m}$, $\vec{H}_0$ is the external magnetic field and $\vec{H}_{(i)}$ stands for the effective field of magnetic anisotropy for i-th sublattices.

Substituting the precession wave solutions with a propagation vector $\vec{k}$ into the system of equations (1) for $i = 1, 2, 3$ and assuming that the order is regular, we obtain formally:

$$R_i(\omega + P^z_{(i)}) = R_i \sum_{j}^{3} S^z_{(j)} b_{ij} + S^z_{(i)} \sum_{j}^{3} R_j f_{ij}; \qquad i = 1, 2, 3 . \tag{2}$$

The R_i denotes the precession radius of spins in the i-th sublattice, and, moreover, the abbreviations

$$b_{ij} = -\frac{2}{\hbar^2} \sum_{\vec{r}}^{i,j} J_{\vec{r}} \quad \text{and} \quad f_{ij}(\vec{k}) = \frac{2}{\hbar^2} \sum_{\vec{r}}^{i,j} J_{\vec{r}} \cos \vec{k}\vec{r}$$

are used. The $\sum_{\vec{r}}^{i,j}$ symbolizes a summation over all lattice vectors $\vec{r}$ beginning in one point of i-th sublattice and pointed to all points of j-th sublattice.

Using the kinematical method we must choose an appropriate coordinate system, rotating with spin. Then, we obtain the same results (2). Thus, both methods appear to be equivalent.

The dispersion law $\omega(\vec{k})$ results as a solution of the secular equation

$$\det\left\{S^z_{(i)} f_{ij} + \delta_{ij}\left[\sum_{\mu}^{3} S^z_{(\mu)} b_{i\mu} - P^z_{(i)} - \omega\right]\right\} = 0 \tag{3}$$

of simultaneous equations (2). If we introduce phase differences for several sublattices, the secular equation (3) remains the same. Hence, the dispersion law is independent of phase difference. From (3) we get three branches of frequency spectrum. For the special case of two kinds of spins, we obtain two branches, namely:

$$\begin{cases} \omega = \pm\tfrac{1}{2}\left[-P^z_{(1)} - P^z_{(2)} + (S^z_{(1)} + S^z_{(2)})(b + f_1)\right] + \\ \qquad \tfrac{1}{2}\left\{\left[P^z_{(2)} - P^z_{(1)} + (S^z_{(1)} - S^z_{(2)})(b - f_1)\right]^2 + 4 S^z_{(1)} S^z_{(2)} f_2^2\right\}^{1/2}. \end{cases} \tag{4}$$

The following abbreviations are used:

$$f_1 = f_{11} + b_{11} = f_{22} + b_{22}; \qquad f_2 = f_{12} = f_{21}; \qquad b = b_{12} = b_{21}.$$

Special cases

In the case of ferrimagnetism with three kinds of spins, it is convenient to specialize the secular equation (3), whereby its solution is simplified. In the case of ferrimagnetism with two kinds of spins, we can use expression (4) for the frequency spectrum; we can specialize it for opposite, as well as for parallel spins. For simplicity, let us put $H_0 = 0$. Thus, in the former case, for nearest neighbours interactions in natural order, neglecting the anisotropy field, we obtain formula (15 . 10) of Kubo ([6], p. 99). For simple cubic lattice, however, expression (4) appears identical with formula (2.7) in the paper of Kouvel and Brooks [5]. Assuming $H_{(i)} = 0$ and small k we find

$$\omega \sim \frac{S^z_{(1)} \cdot S^z_{(2)}}{S^z_{(1)} - S^z_{(2)}} k^2, \tag{5}$$

which is identical with formula (2,9) in [5], and is consistent with formula (31) by Cofta [1] for the ferrimagnetic linear chain. For antiferromagnetism we find easily

$$\omega = \left\{\left[\gamma\mu_0 H_A + S^z(b - f_1)\right]^2 - (S^z f_2)^2\right\}^{1/2}, \tag{6}$$

H_A being the anisotropy field. In approximation, for small k, we obtain from (6) for any natural order the isotropic expression

$$\omega = S^z\{A^2 + B^2k^2\}^{1/2}, \tag{7}$$

and for the non-natural order of MnO, the anisotropic expression [2]

$$\omega = S^z \cdot C \cdot \{k_xk_y + k_yk_z + k_zk_x\}^{1/2} \tag{8}$$

(for simplicity, we have put here $H_A = 0$), A, B and C being constants, depending on lattice structure.

Spin waves in non-regular orders

The semiclassical theory leads to an interesting consequence. Namely along a linear chain of the "non-regular" superstructure of type:

$$\ldots + - - + + - \ldots$$

the wave of spin precession can proceed only if the nearest neighbours are uncoupled. Otherwise, the torque $\vec{S}$ is not perpendicular to the plane formed by $\vec{S}$ and $\vec{\omega}$, as in the precession motion. Thus, there are two ferrimagnetic chains with double lattice constant, mutually uncoupled. The three-dimensional analogon in cubic symmetry would be a body-centered lattice, consisting of two ferromagnetic diamond lattices. Similarly as in the linear case, it must be composed of two uncoupled ferrimagnetic simple cubic lattices.

There exist, of course, intermediate orders, containing the regular (ferromagnetic or ferrimagnetic) as well as the non-regular chains. The propagation of spin precession waves along the non-regular chains in such a superstructure is restricted in the meaning mentioned above. These remarks concern translation lattices only.

The detailed calculations will be published shortly in Acta Physica Polonica.

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On the Classification of Elementary Fermions

by

W. KRÓLIKOWSKI

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Introduction

Considerable attention has been paid recently to the problem of 4-dimensional extension of the isospace [1] – [5]. The aim of this note is to propose a new 4-dimensional isospace, which gathers in a "natural way" leptons and baryons. There is an analogy between this extension and the transition from Pauli's to Dirac's description of the ordinary spin.

We represent the elementary fermions by means of the spinor fields of different ranks in the 4-dimensional isospace and find out the formulae for their electromagnetic charge, mesonic charge and strangeness. Assuming the Pauli principle for the set of isospinor indices of these fields, we obtain a limited number of possible elementary fermions: among them all presently known and some new. Making certain additional assumptions we obtain a theory, in which only the presently known fermions are possible (cf. [6]).

Theory

Let us assume that all elementary fermions may be described by one field $F(x)$ being a 4-component spinor field in the ordinary Minkowski space (here the ordinary spinor index will always be suppressed). We assume further that the field $F(x)$ may be represented as follows ("Fock representation" of the fermion field $F(x)$):

$$F(x) = \begin{pmatrix} F_0(x) \\ F_\alpha(x) \\ F_{\alpha_1\alpha_2}(x) \\ F_{\alpha_1\alpha_2\alpha_3}(x) \\ F_{\alpha_1\alpha_2\alpha_3\alpha_4}(x) \\ \cdots\cdots \end{pmatrix}, \tag{1}$$

where α-indices are 4-component isospin indices in the 4-dimensional isospace with Minkowski metric. Let us assume that the sets of α-indices obey the Fermi-Dirac statistics (Pauli principle). Then, column (1) is limited to its first five components.

The number of α-indices of a component of the field (1) is here called the attribute A of elementary fermions corresponding to this component. We define their strangeness as $S = A - 1$. (For elementary antifermions we take the attribute and the strangeness with the minus sign). We define the total attribute A of a system of elementary fermions as the sum of attributes of the participating fermions, and the total strangeness by the formula

$$S = A - N , \tag{2}$$

where N is the total number of fermions (fermions minus antifermions). The elementary fermions corresponding to components (1) have the attribute $A = 0, 1, 2, 3, 4$ and the strangeness $S = -1, 0, 1, 2, 3$, respectively.

The irreducible representations $D(I_1, I_2)$ of the Lorentz group in the 4-dimensional isospace are numerated by eigenvalues of squared operators, $\vec{I}_1$ and $\vec{I}_2$ being the commutating generators of two 3-dimensional rotation groups. We call the generator $\vec{I}_1$ the total isospin $\vec{T}$. We denote the generator $\vec{I}_2$ by $\vec{\Omega}$. (Then, $D(I_1, I_2) = D(T, \Omega)$). Now, we define the total electromagnetic charge Q and the total mesonic charge G by the formulae

$$Q = e(T_3 + \Omega_3) \tag{3}$$

and

$$G = g\left(\Omega_3 + \frac{A}{2}\right), \tag{4}$$

where A is the total attribute.

Now, we can carry out the reduction of components (1) to the irreducible forms with respect to the Lorentz group in the isospace, and perform the tentative identification of the resulting fields. This reduction procedure consists, of course, in the transition from the "Fock representation" (1) of the field $F(x)$ to another representation, in which $\vec{T}^2, \vec{\Omega}^2, T_3, \Omega_3$ are diagonal. The results are given in Table I. Fermion fields other than those in Table I do not exist, because of the antisymmetry of components (1) with respect to α-indices (Pauli principle).

We can see from Table I that in this theory proton, neutron, neutrino and electron form one 4-component isospinor. The components $\alpha = 1, 2, 3, 4$ of this isospinor correspond to eigenvalues of (τ_3, ω_3) equal to $(+1, +1), (-1, +1), (+1, -1), (-1, -1)$, respectively, where τ_3 and ω_3 are third components of two sets $\vec{\tau}$ and $\vec{\omega}$ of commuting Pauli

matrices, being analogues of the $\vec{\sigma}$ and $\vec{\varrho}$ matrices occurring in the Dirac theory of the ordinary spin.

TABLE I

A	T	Ω	T_3	Ω_3	Q	G	Identification
0	0	0	0	0	0	0	? (second ν)
1	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+e$	g	p^+
			$-\frac{1}{2}$	$+\frac{1}{2}$	0	g	n
			$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	ν
			$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	0	e^-
							$= F_a$
2	0	1	0	$+1$	$+e$	$2g$	?
			0	0	0	g	Λ°
			0	-1	$-e$	0	μ^-
	1	0	$+1$	0	$+e$	g	Σ^+
			0	0	0	g	Σ°
			-1	0	$-e$	g	Σ^-
	0	0	0	0	0	g	? (second Λ°)
3	$\frac{1}{2}$	$\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$+e$	$2g$	?
			$-\frac{1}{2}$	$+\frac{1}{2}$	0	$2g$	?
			$+\frac{1}{2}$	$-\frac{1}{2}$	0	g	Ξ°
			$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	g	Ξ^-
4	0	0	0	0	0	$2g$	?

The operators $A, \vec{T}, \vec{\Omega}$ can be written explicitly, if we introduce the creation and annihilation operators, ξ_a^* and ξ_a (acting on the field $F(x)$), constructing the "Fock representation" (1),

$$
\begin{aligned}
&\langle 0| \\
&\langle a| = \langle 0|\xi_a \\
&\langle a_1 a_2| = \langle 0|\xi_{a_1}\xi_{a_2} \\
&\langle a_1 a_2 a_3| = \langle 0|\xi_{a_1}\xi_{a_2}\xi_{a_3} \\
&\langle a_1 a_2 a_3 a_4| = \langle 0|\xi_{a_1}\xi_{a_2}\xi_{a_3}\xi_{a_4}\,,
\end{aligned} \tag{5}
$$

where $\xi_a (a = 1, 2, 3, 4)$ satisfy the anticommutation relations

$$\{\xi_a, \xi_\beta^*\} = (\delta_4)_{\alpha\beta}\,. \tag{6}$$

Here, the asterisk denotes the Hermitian conjugation and the martix

$$\delta_4 = \omega_3 = \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{7}$$

is the analogue of matrix $\gamma_4 = \varrho_3 = \beta$ in the Dirac theory. Then, we obtain the following formulae

$$A = \int d_3x F^*(x) \bar{\xi}_\alpha \xi_\alpha F(x) , \tag{8}$$

$$\vec{T} = \tfrac{1}{2} \int d_3x F^*(x) \bar{\xi}_\alpha \vec{\tau}_{\alpha\beta} \xi_\beta F(x) , \tag{9}$$

$$\vec{\Omega} = \tfrac{1}{2} \int d_3x F^*(x) \bar{\xi}_\alpha \vec{\omega}_{\alpha\beta} \xi_\beta F(x) , \tag{10}$$

where $\bar{\xi}_\alpha = \xi^*_\beta (\delta_4)_{\beta\alpha}$.

Let us remark that the operators ξ_α (and ξ^*_α) play a similar role in the strong interaction of fermions with K mesons as the operators τ_i in the strong interaction with π mesons (cf. 6]).

As shown in Table I, our theory leads to the existence of all presently known elementary fermions and six new ones. Two of these new fermions have similar physical properties as the neutrino and the Λ^0 particle. Thus, there exist in this theory four essentially new fermions. These four new fermions have double mesonic charges $(2g)$ and, therefore, considerable masses (but ordinary spin $\frac{1}{2}$). We might call them superbaryons.

It is possible to exclude all these new fermions, if it is assumed that

1) the "vacuum element" $\langle 0|$ of base (5) does not correspond to a fermion, maybe because of a connection between the ordinary and isotopic spins; the role of the fundamental element of the base (5) is then played by the element $\langle \alpha| = \langle 0| \xi_\alpha$ $(\alpha = 1, 2, 3, 4)$;

2) the higher elements of base (5) are created not by operators ξ_α, but by operators

$$\eta_\alpha = \tfrac{1}{2}(1 - \omega_3)_{\alpha\beta} \xi_\beta = \begin{vmatrix} 0 \\ 0 \\ \xi_3 \\ \xi_4 \end{vmatrix} , \tag{11}$$

i. e. we have the base

$$\begin{aligned} &\langle \alpha| \\ &\langle \alpha\alpha_1| = \langle \alpha| \eta_{\alpha_1} \\ &\langle \alpha\alpha_1\alpha_2| = \langle \alpha| \eta_{\alpha_1}\eta_{\alpha_2} \end{aligned} \tag{12}$$

and, accordingly, the "Fock representation"

$$F(x) = \begin{pmatrix} F_\alpha(x) \\ F_{\alpha\alpha_1}(x) \\ F_{\alpha\alpha_1\alpha_2}(x) \end{pmatrix}$$

(here $A = 1, 2, 3$ and $S = 0, 1, 2$, respectively).

The reduction performed under these assumptions leads to the results listed in Table II.

TABLE II

A	T	T_3	Ω_3	Q	G	Identification
1	$\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	e	g	p^+
		$-\frac{1}{2}$	$+\frac{1}{2}$	0	g	n
		$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	ν
		$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	0	e^-
						$= F_\alpha$
2	0	0	0	0	g	Λ°
		0	-1	$-e$	0	μ^-
	1	1	0	e	g	Σ^+
		0	0	0	g	Σ°
		-1	0	$-e$	g	Σ^-
3	$\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	g	Ξ°
		$-\frac{1}{2}$	$-\frac{1}{2}$	$-e$	g	Ξ^-

Table II covers exactly the list of elementary fermions presently known (cf. [6]).

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Angular Distribution of High-energy Bremsstrahlung

by

A. DELOFF and R. GAJEWSKI

Presented by W. RUBINOWICZ on June 7, 1958

Angular distribution of Bremsstrahlung for sufficiently large angles of observation was obtained by Hough [1]. However, his formula fails in the vicinity of the high-energy end of the radiation spectrum. More recently [2] it has been recognized that mesurements in the region of the high-energy tip of the Bremsstrahlung can provide valuable physical information, and an accurate experimental method has been developed (cf. [3]), based on the scattering of the Bremsstrahlung quanta on the 15.11 MeV level of ^{12}C. In this paper, we discuss angular and energy distribution of Bremsstrahlung quanta for large angles and at the high-energy tip of the spectrum.

We start from the well known Bethe – Heitler formula and use the notations of [4]. The influence of electron screening can be neglected, as the impact parameter for large scattering angles is small. The influence of charge distribution in the nucleus is accounted for by introducing the nuclear form factor $F(q)$ defined by (see [1], [5])

$$F(q) = \frac{4\pi}{Ze} \int_0^\infty \varrho(r) \frac{\sin qr}{qr} r^2 dr \,, \tag{1}$$

where $\varrho(r)$ is the density of the nuclear charge. We note that, given the energy of the quantum and the angle of observation or, in other words, given vector $\boldsymbol{k}$, and given also the momentum of the incident electron $\boldsymbol{p}_0$, the quantity q defined by $q = |\boldsymbol{p}_0 - \boldsymbol{p} - \boldsymbol{k}|$ can take values between $q_{\min} = |\boldsymbol{p}_0 - \boldsymbol{k}| - p$ and $q_{\max} = |\boldsymbol{p}_0 - \boldsymbol{k}| + p$. We can express this fact by writing

$$\underset{\min}{q_{\max}} = q_0 \pm \Delta q \,, \tag{2}$$

$$q_0 \equiv |p_0 - k| = [E_0^2 - 1 + k^2 - 2(E_0^2 - 1)^{1/2} k \cos\theta_0]^{1/2} \,, \tag{3}$$

$$\Delta q = p = (\varepsilon^2 + 2\varepsilon)^{1/2} \,, \tag{4}$$

$$E_0^2 = p_0^2 + 1; \quad E^2 = p^2 + 1; \quad \varepsilon = E - 1; \quad E_0 = 1 + \varepsilon + k \,. \tag{5}$$

For values of ε, k and θ_0 such that $\Delta q/q_0 \ll 1$, it is possible to replace q by q_0 in the Bethe – Heitler formula and in (1). Then, integration over the angles θ, φ yields

$$\bar{\Phi}^{-1} d\varphi/(dk\, d\omega) = F(q_0) S(k, \theta_0, \varepsilon) , \tag{6}$$

where

$$S(k, \theta_0, \varepsilon) = (\pi k E_0)^{-1} q_0^{-4} p \{2(4E_0^2 - q_0^2)[(E/p)\ln(E+p) - 1] + (4E^2 - q_0'^2)\operatorname{ctg}^2 \tfrac{1}{2}\theta_0 + [Ek^2/(E_0 \sin^2 \tfrac{1}{2}\theta_0)][1 + (Ep)^{-1}\ln(E+p)]\}, \tag{7}$$

$$\bar{\Phi} = \frac{Z^2 e^4}{137 \mu^2} , \qquad d\omega = 2\pi \sin\theta_0 d\theta_0 .$$

Because of the factor q_0^{-4}, the error arising from such an approximation will not exceed $4\Delta q/q_0$. The range of ε, k and θ_0 for which this error is kept at a reasonable level can be estimated by noting that, with $k \gg 1$, and $k \gg \varepsilon$, Eq. (3) leads to $q_0 \approx 2k \sin \frac{1}{2}\theta_0$ (cf. [5]), so that one can assume

$$\Delta q/q_0 = \tfrac{1}{2}(\varepsilon^2 + 2\varepsilon)^{1/2}/(k \sin \tfrac{1}{2}\theta_0) .$$

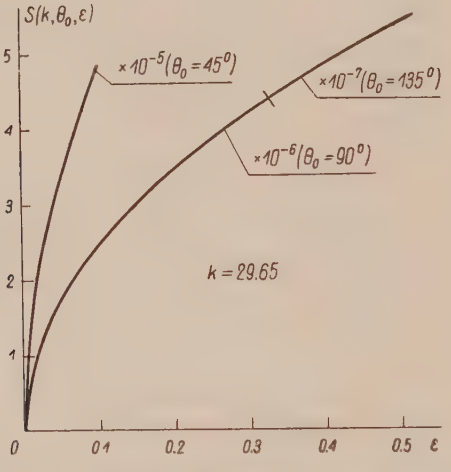

Fig. 1. Function $S(k, \theta_0, \varepsilon)$ of Eq. (6) plotted against $\varepsilon = E_0 - 1 - k$ for $k = 29.65$ (15.12 MeV) and $\theta_0 = 45°, 90°, 135°$. Curves for 90° and 135° coincide. At the end of each curve maximum error $4\Delta q/q_0$ is equal to 10%. The end of the 90° curve is marked by a dash

In Fig. 1 function (7) is plotted against ε for $k = 29.65$ which corresponds to 15.12 MeV photons of [3], and for angles of observation $\theta_0 = 45°, 90°, 135°$. At the end of each curve there is $4\Delta q/q_0 \approx 10\%$.

For relatively small values of q_0, the form factor $F(q_0)$ depends weakly on q_0, and even more weakly on ε. Thus, to obtain the actual cross-section, each of the curves, Fig. 1, should be multiplied by a numerical constant of order unity: $F(q_0) \approx F(2k \sin \frac{1}{2}\theta_0)$.

For small values of $\varepsilon(\varepsilon \ll 1)$ formula (7) can be reduced to

$$S(k, \theta_0, \varepsilon) = f(k, \theta_0)(2\varepsilon)^{1/2}, \tag{8}$$

where

$$f(k, \theta_0) = (\pi k E_0)^{-1} q_0^{-4}[(4 - q_0^2)\operatorname{ctg}^2 \tfrac{1}{2}\theta_0 + 4E_0 k \cos^2 \tfrac{1}{2}\theta_0], \tag{9}$$

with $E_0 = 1 + k$. In Fig. 2 function $f(k, \theta_0)$ is plotted against k for $\theta_0 = 45°, 90°, 135°$. It allows to predict immediately the order of magnitude of the relative cross-section (6) for small values of ε. If $k \gg 1$, we have approximately

$$f(k, \theta_0) \approx [4\pi k^2(1 + k)]^{-1} \operatorname{ctg}^2 \tfrac{1}{2}\theta_0 / \sin^2 \tfrac{1}{2}\theta_0, \tag{9'}$$

which is shown by the dotted line in Fig. 2.

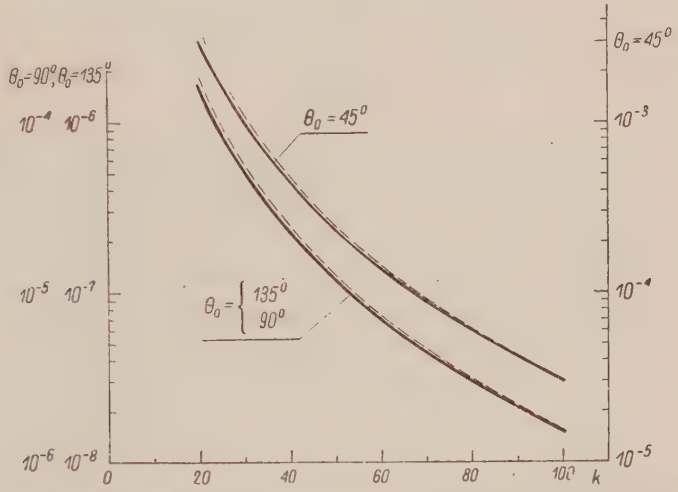

Fig. 2. Function $f(k, \theta_0)$ of Eq. (8) plotted against k for $\theta_0 = 45°, 90°, 135°$. Curves for 90° and 135° coincide. Full line gives $f(k, \theta_0)$ as calculated from (9), dotted line represents approximate formula (9')

We are indebted to Dr J. Werle for a valuable discussion.

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On the Effect of Concentration on the Polarization of the Fluorescence of Rigid Solutions

by

A. KAWSKI

Presented by A. JABŁOŃSKI on March 12, 1958

Introduction

Concentrational depolarization of the photoluminescence of solutions of dyes was first observed in 1924, [1], [2], [3]. It was possible to determine the quantitative relationship between the depolarization, the concentration and the mean lifetime τ of the excited state of the fluorescence only after methods had been elaborated [4] by which concentrational depolarization could be treated separately from depolarization resulting from long range reabsorption and re-emission. If the layer of the solution chosen is sufficiently thin, the latter effects may be practically eliminated. Measurements of the concentrational depolarization were made by Y. Cauchois [5] for fluorescein and by Feofilov and Sveshnikov [6] for fluorescein, eosin, rhodamin *B* and trypaflavine in glycerol. In connection with measurements by Feofilov and Sveshnikov, theories of concentrational depolarization were published by Vavilov [7], Förster [8], and recently, by Weber [9] and Jabłoński [10].

Jabłoński assumes fluctuations of concentration of the luminescent molecules and migration of the excitation energy within the "active sphere"; migration of the energy is assumed to be the cause of the self-depolarization of the photoluminescence. Basing partly on these assumptions, Jabłoński obtained a general relationship for the degree of polarization as function of the concentration, which, if the term accounting for self-quenching be omitted, takes the form

$$p = 3p_0 \frac{\sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!} \cdot \frac{2}{k+1}}{\sum_{k=1}^{\infty} \frac{\nu^{k-1}}{(k-1)!} \left[(3-p_0) + \frac{2p_0}{1+k} \right]}, \tag{1}$$

where $v = v \cdot n$, n denoting the number of luminescent molecules per cm³, v – the volume of the active sphere, and p_0 – the fundamental degree of polarization.

Eq. (1) may be rewritten more conveniently as follows [11]:

$$\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_0} \cdot \frac{3 - p_0}{3} f(v), \tag{2}$$

where

$$f(v) = \frac{v^2}{2(v - 1 + e^{-v})} - 1.$$

If $v \ll 1$, Eq. (2) yields

$$\frac{1}{p} \approx \frac{1}{p_0} + \frac{1}{p_0} \cdot \frac{3 - p_0}{3} \cdot \frac{1}{3} v. \tag{3}$$

It is the aim of the present investigation to test the above results of Jabłoński's theory for the case of fluorescent rigid solutions.

Experimental part

a) Method of preparation of rigid solutions of fluorescent substance. Rigid solutions of fluorescent substances were obtained by the process of polymerization of a solution of anthracene and fluorescein in methyl methacrylate monomer.

The rigid solutions were obtained as follows: solutions of various concentrations of anthracene and fluorescein in methyl methacrylate monomer were prepared and polymerized in hermetically sealed flasks at c. 80°C. After several hours, the liquid having evolved through syrop-like consistency to that of a jelly, the contents of the flasks were transferred to plane-parallel vessels prepared for the purpose, in order to obtain plates of a known thickness, and, on sealing the vessels hermetically, polymerization was proceeded with at room temperature. Very thin layers were obtained by letting the jelly spread over the glass surface of a flat container. During the process of polymerization special care was taken to keep a constant temperature, in order to avoid stress in the samples.

Fluorescein was found to have a very low solubility in methyl methacrylate; if, however, a solution of fluorescein in ethyl alcohol is prepared and then added to methyl methacrylate, quite considerable concentrations amounting to $5 \cdot 10^{-3}$ g./cm.³ are obtained. An alternative is first to add 1 volume-unit of alcohol to 9 of methyl methacrylate, and subsequently to dissolve the fluorescein in the mixture. An addition of as much as 3 volume-units of ethyl alcohol to 10 of methyl methacrylate was found to be in no way detrimental to the process of polymerization. In evaluating the concentration of the fluorescent substance in the rigid solution (plexiglass) the coefficient of contraction was accounted for too.

b) Method of measurement. The polarization of the luminescence of rigid solutions of anthracene and fluorescein in plexiglass was measured by the well-known visual compensation method, using a Savart polariscope and an Arago compensator.

In order to eliminate the effect of scattered light, the fluorescence was excited with light which had passed through a Wood filter (mercury lamp and filter transmitting the 366 mμ line in the case of fluorescein, and the 366 mμ together with the less intensive 334 mμ and 313 mμ lines in that of anthracene). The exciting light was polarized with a Glan polarizing prism. The interference structure produced by the Savart polariscope was projected upon the wide open slit of a Hilger spectrograph [12] in which the camera could be eventually replaced by a telescope; by using a slit arrangement therein a sector of the fluorescence spectrum could be conveniently separated. Using the Arago compensator, the Savart fringes visible on the fluorescence band were extinguished. Sometimes, observation was conducted through an appropriate filter; in all cases, it was at a small angle with regard to the exciting light pencil.

The degree of polarization was computed from formulae [13] for 2 and 4 glass plates; this number was raised to 6 if the degree of polarization exceeded 0.40. In that case there is a sharp limit of appearance and disappearance of Savart fringes. On the other hand, an augmentation of the number of plates considerably reduces the angle through which the set of glass plates has to be turned in order to bring the structure to extinction. As already stated [14], for small angles the formulae yield the degree of polarization with exactitude. Thus, the number of glass plates should be so chosen as to avoid exceeding, in the process of compensation, the angle $\alpha = 60°$ at which the compensator is set.

In order to eliminate the effect of secondary fluorescence, the measurements were made with samples having the shape of plates, the thickness of which was appropriately chosen.

Results and discussion

a) Anthracene in plexiglass. For anthracene in plexiglass, the curve of concentrational depolarization was obtained (Fig. 1), the ordinates showing the degree of polarization, and the abscissae — the concentration expressed as the number of luminescent molecules per cm.^{3} The fundamental degree of polarization of anthracene in plexiglass was measured, yielding, for the concentration $c \sim 10^{-5}$ g./cm.^{3}, the value of 0.257 (with the exciting wave-length $\lambda = 366$ mμ and the Hg lines of lesser intensity 334 and 313 mμ). It should be of interest to compare the degree of fundamental polarization obtained by the present author with the values obtained by others. Perrin (1929) [15], who investigated a rigid solution of anthracene in Canada balm excited with ultraviolet

radiation, found $p_0 = 0.26$, whereas, for an anthracene solution in vitrous ethyl alcohol at the temperature of $-180°$C., he gives $p_0 = 0.25$ (the concentration is not known). Spektorov (1949) [16], basing on measurements of the degree of polarization versus the viscosity of the solution $\left[\frac{1}{p} = f\left(\frac{T}{\eta}\right)\right]$ obtained $p_0 = 0.263$ for anthracene in glycerol (with an addition of ethyl alcohol), at a concentration of $c = 10^{-5}$ g./cm.^{3} and with an exciting wave-length of 365 mμ.

It will be noted that the above values are in very good agreement. Chakravarti and Ganguly (1941) [17], however, when investigating anthracene in anhydrous glycerol at $-79°$C., found a fundamental degree of polarization amounting to $p_0 = 0.44$, a value considerably different from those given above.

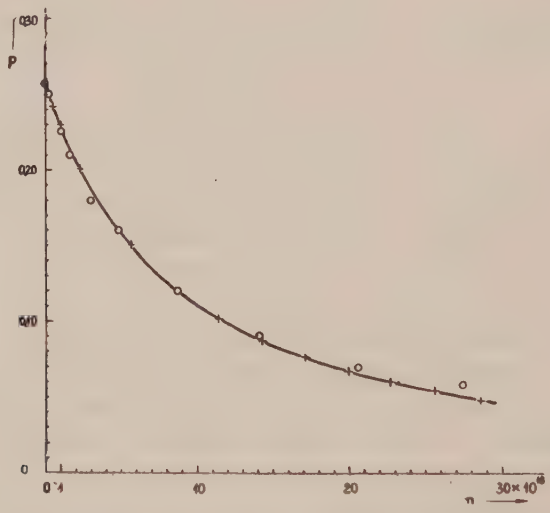

Fig. 1. Dependence of degree of polarization of the fluorescence of anthracene in polymethyl methacrylate on the number of luminescent molecules per cm.^{3}

○ — measured
\+ — calculated from Eq. (2)

The shape of the concentrational depolarization curve for anthracene is described with accuracy by Jabłoński's theory, which has already been discussed in the introduction to the present paper, and in that of Bojarski [11]. Eq. (2) contains the volume of the "active sphere", which is determined experimentally. In Fig. 2 the ordinates represent the reciprocal of the degree of polarization $1/p$, and the abscissae — $v = v \cdot n$, the theoretical values having been traced from Eq. (2) for the fundamental degree of polarization $p_0 = 0.257$. The same graph is then used for determining v for each value of the concentration and the respective measured degree of polarization. Since $v = v \cdot n$, the radii of the "active spheres" corresponding to each value of n (the number of molecules

per cm.[3]) may be computed, yielding the mean radius of the active sphere: $R = 44$ Å.

Within the range of the concentrations investigated, the radius of the active sphere is practically independent of the concentration, a fact that was recognized by Frąckowiak [18], who determined the radius of the sphere from the experimental decay curves. With the value of the radius of the active sphere thus obtained, those of v corresponding to the experimental points were then determined. In Figs. 1 and 2 the ex-

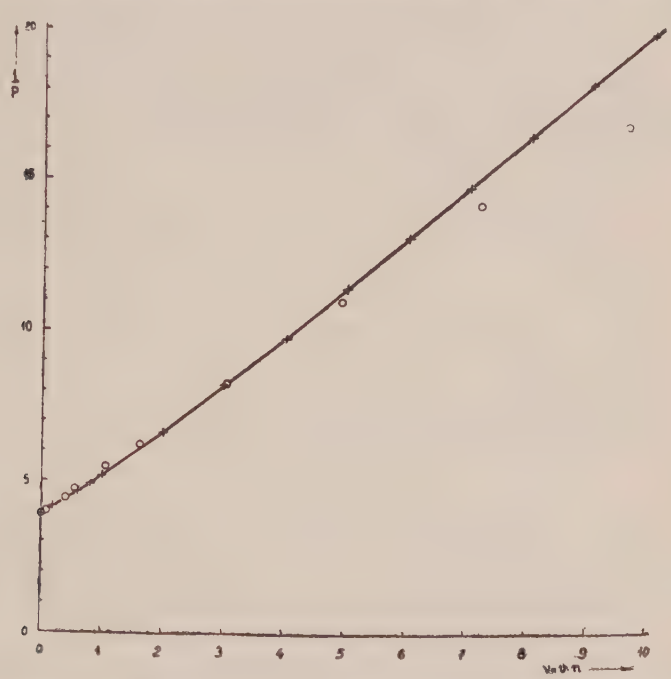

Fig. 2. Reciprocal of the degree of polarization of the fluorescence of anthracene in polymethyl methacrylate as function of $v = v \cdot n$.
o — measured
+ — calculated from Eq. (2)

perimental values and the theoretical results corresponding to $R = 44$ Å are compared. Expression (2) fit very well to the experimental results for not too high concentrations of anthracene in plexiglass. The last two points for the highest concentrations deviate from the calculated curve (Fig. 2). This deviation may be due to the neglected effect of selfquenching.

b) Fluorescein in plexiglass. For a rigid solution of fluorescein in plexiglass (containing alcohol) and the exciting Hg wave-length of 366 mμ, no decrease in the polarization with growing concentrations was found within a very broad range of the latter ($3 \cdot 10^{-6}$-$6{,}6 \cdot 10^{-3}$ g./cm.[3]), (viz. Fig. 3). Concentrational depolarization may take place at much higher concentrations of fluorescein in plexiglass than is the case with

glycerol solutions. However, it was not possible to exceed the concentration of $6.6 \cdot 10^{-3}$ g./cm.^{3} Experiments were also made with the aim of obtaining concentrational depolarization with exciting light of longer wave-lengths, namely, $\lambda \sim 436$ mμ and $\lambda \sim 490$ mμ; these yielded the same function of the degree of polarization as that obtained with 366 mμ exciting wave-length. The values of the fundamental degree of polarization of fluorescein in plexiglass excited with light of $\lambda \sim 436$ mμ and $\lambda \sim 490$ mμ were: $p_0 = 0.427$ and $p_0 = 0.448$, respectively. However, the exciting wave-length $\lambda = 366$ mμ yielded $p_0 = 0.38$. The lack of any decrease in polarization with growing concentration of fluorescein in plexiglass is probably connected with the rapid decrease in the duration of the luminescence resulting from concentrational quenching. The effect is now under investigation.

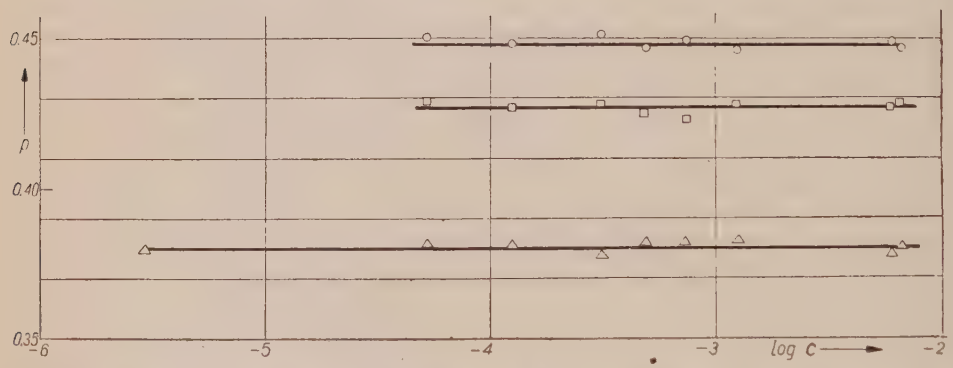

Fig. 3. Dependence of degree of polarization of the fluorescence of fluorescein in polymethyl methacrylate on the concentration (in g./cm.^{3})
○ — measured with exciting wave-length $\lambda \sim 490$ mμ,
□ — measured with exciting wave-length $\lambda \sim 436$ mμ,
△ — measured with exciting wave-length $\lambda = 366$ mμ.

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БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VI 1958 ВЫПУСК 8

М. ФИШ. ПРЕДЕЛЬНАЯ ТЕОРЕМА ДЛЯ НЕУБЫВАЮЩИХ СЛУЧАЙНЫХ ФУНКЦИЙ . . . . . стр. 485–487

Обобщая теорему Гливенки [2], автор доказывает следующую теорему.

Теорема. *Пусть* $\{Y_k(t), -\infty < t < \infty\}$ $(k = 1, 2, 3 \ldots)$ — *последовательность действительных, сепарабельных, независимых и одинаково распределенных стохастических процессов. Если соотношение* $P\{Y_k(t_1) \leqslant Y_k(t_2)\} = 1$ *имеет место для любых* $t_1, t_2 (t_1 < t_2)$ *и функция* $F(t) = EY_k(t)$ $(-\infty < t < \infty)$ *существует и удовлетворяет соотношениям*

$$F(-\infty) = b\,, \qquad F(+\infty) = c\,,$$

где c *и* b — *некоторые постоянные, тогда*

$$P\left(\lim_{n\to\infty} \sup_t \left| \frac{1}{n} \sum_{k=1}^{n} Y_k(t) - F(t) \right| = 0\right) = 1\,.$$

А. ЯНКОВСКИЙ, АКСИОМАТИКА ГЕОМЕТРИИ МЭБИУСА стр. 489–494

В работе приводится определение геометрии Мэбиуса и ее модель в метрической эвклидовой геометрии. Эта модель состоит из:

1. Множества точек n-мерного эвклидова пространства с присоединенной одной бесконечно удаленной точкой. Эта бесконечно удаленная точка, по определению, инцидентна со всякой прямой и не инцидентна ни с одной сферой.

2. Множества M-шаров, т. е. множества гиперсфер и гиперплоскостей n-мерного эвклидова пространства.

3. Соотношения инциденции, являющейся расширением обыкновенной инциденции путем присоединения инциденции бесконечно удаленной точки.

Далее автор переходит к рассмотрению своего основного вопроса, где приводится система аксиом, основанная на упомянутых выше трех первоначальных понятиях и доказывает, что эта система аксиом

(1) существенно выполнима в так определенной модели,

(2) категорична.

Из (1) и (2) следует, что эта система аксиом является действительно аксиоматикой геометрии Мэбиуса.

Г. МИЛИЦЕР-ГРУЖЕВСКАЯ, ГРАНИЧНОЕ СВОЙСТВО ОБОБЩЕННОГО ПОТЕНЦИАЛА ПРОСТОГО СЛОЯ ПО ОТНОШЕНИЮ К НОРМАЛЬНОМУ ПАРАБОЛИЧЕСКОМУ УРАВНЕНИЮ . . . . . стр. 495–499

В настоящей работе доказано, что границей потенциала простого слоя [2] по отношению к параболическому уравнению (1) и плотности $\varphi(Q, \tau)$, при $t \to \infty$ является потенциал простого слоя с граничной плотностью $\varphi(Q) = \lim_{\tau \to \infty} \varphi(Q, \tau)$ относительно граничного уравнения (10,1), которое является эллиптическим уравнением, если только выполняются предположения I, II, III, IV, V и VI. Часть этих предположений касается существования решений параболического уравнения (1), часть же касается существования границ плотности и коэффициентов уравнения (1), а также условий, которые должны удовлетворять этим границам. Предположение VI в своей сущности ограничивает применяемость теоремы до потенциалов на поверхностях S, ограничивающих небольшие области в зависимости от величины коэффициентов уравнения (1).

З. ЦЕСЕЛЬСКИЙ, ОБ АБСОЛЮТНОЙ СХОДИМОСТИ РЯДОВ ФУРЬЕ ПОЧТИ ВСЕХ ФУНКЦИЙ ИЗ ПРОСТРАНСТВА ВИНЕРА стр. 501–503

Пусть C — пространство функций $x(t)$ непрерывных в $\langle 0, 1\rangle$ и исчезающих в нуле. В C вводим меру [5], которую называем *мерой Винера*.

Пусть ω — действительная функция, определенная в $\langle 0, \infty\rangle$ и удовлетворяющая условиям: 1° $\omega(0) = 0$, 2° $\omega(t)$ — возрастающая, 3° $\omega(2t)/\omega(t)$ — ограниченная при $t \to \infty$.

Скажем, что функция ω удовлетворяет условию (c), если выполняет условия 1°, 2°, 3° и ряд $(*)$ сходится. Скажем, что функция ω удовлетворяет условию (d), если удовлетворяет условиям 1°, 2°, 3° и ряд $(*)$ расходится.

Через $\{\omega_n\}$ обозначаем одну из трех систем: тригонометрическую, Хаара, Уольша. Система $\{\omega_n\}$ нормирована в $\langle 0, 1\rangle$. Сохраняя обозначения, введенные перед теоремой I, получаем следующие теоремы:

Теорема I. *Пусть* $0 < \alpha < 1 < \beta < 2$. *Если* ω *удовлетворяет условию* (c), *то для почти каждого* $x \in C$ (*кроме множества меры Винера нуль*) *существует* $n[x]$ *натуральное такое, что выполняются условия* (1), (3) *и* (5).

Теорема II. *Пусть* $0 < \alpha < 1 < \beta < 2$. *Если* ω *удовлетворяет условию* (d), *то для почти каждого* $x \in C$ *существует* $n[x]$ *натуральное такое, что выполняются условия* (2), (4) *и* (6).

А. ПИСКОРЭК, О НЕКОТОРЫХ ГРАНИЧНЫХ ПРОБЛЕМАХ ДЛЯ ПОЧТИ ЛИНЕЙНЫХ НОРМАЛЬНЫХ ПАРАБОЛИЧЕСКИХ УРАВНЕНИЙ стр. 505–510

В работе приводится доказательство существования решения нелинейной смешанной задачи (при краевых и начальных условиях) для параболического уравнения вида (1), при краевом нелинейном условии (2) и начальном однородном $(f(X) \equiv 0)$ условии (3), а также неоднородном $(f(X) \not\equiv 0)$. Задача рассматривается для ограниченной области Ω, находящейся в n-мерном эвклидовом пространстве.

Предполагается, что граница области Ω, коэффициенты уравнения (1), функции выступающие в этом уравнении в краевом условии (2) и в начальном условии (3), удовлетворяют соотношениям I., II., 1. III., 1. IV., 2. III., 2. IV., 2. V.

Эту смешанную задачу автор сводит к системе $(n+2)$ интегральных уравнений (5) с $(n+2)$ неизвестными функциями. Затем автор доказывает существование решения этой системы при помощи теоремы о неподвижной точке непрерывной функциональной операции И. Шаудера.

С этой целью автор рассматривает Банахово пространство систем $(n+2)$ непрерывных функций и изучает множество E, определенное реляцией (6).

На этом множестве определяется функциональная операция (7). Если выполняются неравенства (8) и (9), то образ множества E, полученный при помощи этой операции содержится в E. Кроме того, этот образ является компактным множеством, откуда вытекает существование решения системы (5). Следовательно, в случае однородного начального условия, существует решение задачи, поставленной для уравнения (1).

В случае неоднородного начального условия, вместо функциональной операции (7), рассматривается функциональная операция (10), для которой аналогичным образом доказывается существование решения системы (5). Следовательно, в обоих случаях существует решение смешанной задачи для уравнения (1).

А. ЛЕЛЕК, О МНОГОЗНАЧНЫХ ИНВОЛЮЦИЯХ . . . . . стр. 511–517

Функция $F: X \to 2^X$, значениями которой являются замкнутые множества пространства X, называется *полунепрерывной сверху*, если $\lim\limits_{n\to\infty} x_n = x$ влечет за собою $\underset{n\to\infty}{Ls}\, F(x_n) \subset F(x)$.

Пусть наличие в множестве $X - A$, где $A \subset X$, такой компоненты C, которая содержит образ $F(A) = \bigcup\limits_{x\in A} F(x)$ множества A, называется свойством (α) этого множества. Справедлива теорема: *если X — локально связный и уникогерентный континуум, функция $F: X \to 2^X$ полунепрерывна сверху и $G \subset X$ — открытое множество, то или G, или $X - G$ содержит компоненту A, не обладающую свойством (α).*

Функция $\Phi: X \to C(X)$, значениями которой являются подконтинуумы пространства X и которая непрерывна сверху, называется *многозначной инволюцией*, если $x_1 \in \Phi(x_2)$ для всякого $x_2 \in \Phi(x_1)$. Из предыдущей теоремы следует, что *при тех же предположениях, касающихся пространства X, если функция $\Phi: X \to C(X)$ является многозначной инволюцией и если открытое множество $G \subset X$ удовлетворяет условию $\Phi(G) \subset G$, то или G, или $X - G$ содержит компоненту A, равную своему образу $\Phi(A)$.*

Последняя теорема является обобщением известной леммы Куратовского (см. [1], стр. 90 и [3], стр. 117) об обыкновенных, т. е. однозначных, инволюциях.

Г. ЦОФТА, СПЕКТР ЧАСТОТЫ СПИНОВЫХ ВОЛН ДЛЯ РАЗЛИЧНЫХ УПОРЯДОЧЕНИЙ ФЕРРИМАГНИТНЫХ ТРАНСЛЯЦИОННЫХ РЕШЕТОК . . . . . стр. 519—522

В настоящей работе выведена общая дисперсионная формула для частоты спиновой волны в произвольной трансляционной ферримагнитной решетке, образованной из двух или трех типов спинов. При этом были учтены более общие упорядочения (сверхструктуры), чем обычно.

Для выведения этой формулы был использован полуклассический метод. Базирующие на нем два способа выведения: формальный и кинетический оказались эквивалентными.

Полученные формулы учитывают произвольно далеких соседей. Они применяются для каждого „регулярного" упорядочения, т. е. такого, в котором оба спины, расположенные симметрически по отношению к данному спину, принадлежат к одной и той же подрешетке. В отдельных случаях выведенная дисперсионная формула для ферримагнетика с двумя типами спинов переходит в выражения, известные из других работ.

В упорядочениях не регулярных, полуклассический метод предсказывает ограничения распространения спиновых волн.

В. КРУЛИКОВСКИЙ, О КЛАССИФИКАЦИИ ЭЛЕМЕНТАРНЫХ ФЕРМИОНОВ . . . . . стр. 523—527

В настоящей работе предлагается четырехмерное расширение пространства изотопного спина, соединяющее „естественным образом" лептоны и барионы. Это расширение является аналогией перехода от описания обыкновенного спина Паулем к описанию того же спина Дираком.

Принимая некоторые предположения, получено ограниченное число возможных элементарных фермионов, среди которых находятся все известные в настоящее время фермионы.

А. ДЕЛОФФ и Р. ГАЕВСКИЙ, УГЛОВОЕ РАСПРЕДЕЛЕНИЕ ТОРМОЗНОГО ИЗЛУЧЕНИЯ ВЫСОКОЙ ЭНЕРГИИ . . . . . стр. 529—531

На основании формулы Бэтэ-Гайтлера было найдено дифференциальное эффективное сечение на тормозное излучение электронов, рассеиваемых ядрами для высокоэнергетического края спектра и больших углов наблюдения фотонов.

А. КАВСКИЙ, ВЛИЯНИЕ КОНЦЕНТРАЦИИ НА ПОЛЯРИЗАЦИЮ ФЛУОРЕСЦЕНЦИИ ЖЕСТКИХ РАСТВОРОВ . . . . . стр. 533—539

Исследована зависимость степени поляризации от концентрации для антрацена и флуоросценция в жестком растворе метакрилана метила.

Снижение степени поляризации с ростом концентрации антрацена в плексигласе можно вполне хорошо описать пользуясь теорией Яблонского, касающейся самодеполяризации. В случае флуоресцеина в плексигласе для довольно большого предела концентраций, не обнаружено снижение степени поляризации с ростом концентрации для волны длиной $\lambda = 366\ м\mu$, для $\lambda \sim 436\ м\mu$, а также для $\lambda \sim 490\ м\mu$.

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